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Longest increasing subsequences, Plancherel-type measure and the Hecke insertion algorithm

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ABSTRACT

We define and study the *Plancherel–Hecke probability measure* on Young diagrams; the *Hecke algorithm* of Buch–Kresch–Shimozono–Tamvakis–Yong is interpreted as a polynomial-time exact sampling algorithm for this measure. Using the results of Thomas–Yong on *jeu de taquin* for *increasing tableaux*, a symmetry property of the Hecke algorithm is proved, in terms of longest strictly increasing/decreasing subsequences of words. This parallels classical theorems of Schensted and of Knuth, respectively, on the *Schensted* and *Robinson–Schensted–Knuth* algorithms. We investigate, and conjecture about, the limit typical shape of the measure, in analogy with work of Vershik–Kerov, Logan–Shepp and others on the "longest increasing subsequence problem" for permutations. We also include a related extension of Aldous–Diaconis on *patience sorting*. Together, these results provide a new rationale for the study of increasing tableau combinatorics, distinct from the original algebraic-geometric ones concerning K-theoretic Schubert calculus.

1. Introduction and main results

1.1. Overview

Let $W_{n,q}$ denote the set of words of length n in the alphabet $\{1, 2, \ldots, q\}$. Let $\mathtt{LIS}(w)$ denote the **length of the longest strictly increasing subsequence** of $w = w_1 w_2 \cdots w_n$, i.e., the largest ℓ with a subsequence $i_1 < i_2 < \cdots < i_\ell$ such that $w_{i_1} < w_{i_2} < \cdots < w_{i_\ell}$. Similarly, we consider the **length of the longest strictly decreasing subsequence** $\mathtt{LDS}(w)$ of w. Our main goal is to introduce and study

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a discrete probability measure on Young diagrams, in connection with the study of the distributions of LIS and LDS on uniform random words. An additional goal is to provide a novel motivation for the K-theoretic Schubert calculus combinatorics of [7,24].

There are analogies with the study of LIS and LDS in the **permutation case**, i.e., when w is chosen uniformly at random from the symmetric group S_n. The latter topic has attracted considerable attention; we refer the reader to the surveys [1,23] and the references therein. In the permutation case, random Young diagrams are distributed according to the *Plancherel measure* (on irreducible representations) of S_n. This discrete probability measure is the push-forward of the uniform distribution on S_n, under the *Robinson–Schensted correspondence*. Schensted [20] established that this correspondence encodes $\mathtt{LIS}(w)$ and $\mathtt{LDS}(w)$ symmetrically in the shape λ associated to w. In [26,18], these ideas are applied to determine the asymptotics of the expectation of LIS over S_n (solving the old "longest increasing sequences problem"), via a study of the "limit typical shape" under the Plancherel measure.

As a continuation of this theme, we apply the Hecke (insertion) algorithm of [7] to define the Young diagram $\mathtt{Heckeshape}(w)$ for each $w \in W_{n,q}$; using this we define *Plancherel–Hecke measure*. Our belief that this measure should actually be worthy of analysis was initially guided by our theorem that Hecke symmetrically encodes $\mathtt{LIS}(w)$ and $\mathtt{LDS}(w)$ for $w \in W_{n,q}$, a generalization of Schensted's theorem. During the course of our investigation, we found that many other aspects of the Plancherel–Hecke measure (conjecturally) also resemble those of the Plancherel measure. This paper records these results, both theoretical and computational, as a justification for further study.

Briefly, this is how the two aforementioned measures compare. We consider the behaviour as n goes to infinity, and q grows proportionally to n^α for a fixed α. We conjecture that for $\alpha > \frac{1}{2}$, our measure is concentrated around the limit typical shape under Plancherel measure. This *Plancherel curve* plays an important role in [26,18]. On the other hand, for $\alpha < \frac{1}{2}$ we conjecture the measure is concentrated near the "staircase shape". In particular, a "phase transition" is suggested at $\alpha = \frac{1}{2}$. As we tune α, a symmetric deformation of the Plancherel curve occurs. In view of the above mentioned result on the Hecke algorithm, this transition phenomenon is further evidenced by computations (with contributions by O. Zeitouni) of the expectation of LIS and LDS as α varies; see Section 5 and Appendix A.

There have been earlier extensions of the permutation case to $W_{n,q}$. The limit distribution of the length of the longest *weakly* increasing/decreasing subsequence (LwIS/LwDS) on $W_{n,q}$ was found in work of [25], following the breakthrough [2] on the limit distribution of LIS on S_n. See also the more recent work [14]. However, analogous understanding of the distribution of LIS and LDS on $W_{n,q}$ appears to be less developed; see, e.g., [3,5,25] for contributions.

As a point of comparison and contrast with our approach, previous work on LIS, LDS and $W_{n,q}$ utilizes the combinatorics of the *Robinson–Schensted–Knuth correspondence*, which asymmetrically encodes LwIS and LDS. We offer an alternative viewpoint on the relationship between Young diagrams and LIS, LDS. New questions and conjectures are raised, stemming from the Coxeter-theoretic viewpoint of [7] (which in turn generalizes ideas of [9]).

This text expresses our desire to point out a natural link between the probabilistic combinatorics of LIS, LDS and the combinatorial algebraic geometry of K-theoretic Schubert calculus. In particular, we apply and further develop the *jeu de taquin* for *increasing tableaux* from [24], thereby giving another perspective on that work, distinct from the original one. In summary, we believe that the availability of these two disparate interpretations for [7,24] provides something atypical to recommend K-theoretic tableau combinatorics, among the large array of interesting generalizations of the classical Young tableau and symmetric function theories known today.

1.2. Plancherel–Hecke measure

We identify a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0)$ with its Young diagram (in English notation); set $|\lambda| := \sum_i \lambda_i$. Let $\mathbb{Y}$ denote the set of all Young diagrams. A **filling** is an assignment of a label from $\{1, 2, \ldots, q\}$ to each box of the Young diagram λ. A filling is an **increasing tableau** if it is strictly increasing in both rows and columns. Let $\mathtt{INC}(\lambda, q)$ be the set of all increasing tableaux of shape λ.

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \in \texttt{INC}((4,2,1),5) \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 1,2 & 4 & 5,7,9 & 10 \\ \hline 3,6 & 11 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \in \texttt{SsetT}((4,2,1),11).$$

Fig. 1. An increasing tableau and a set-valued standard Young tableau.

We also need **set-valued tableaux** [6], which are fillings of λ assigning to each box a nonempty subset of $\{1, 2, \ldots, n\}$ such that the largest entry of a box is smaller than the smallest entry in the boxes directly to the right of it, and directly below it. We call a set-valued tableau **standard** if each $i \in \{1, \ldots, n\}$ is used exactly once. Let $\texttt{SsetT}(\lambda, n)$ denote the set of all standard set-valued tableaux. See Fig. 1.

The **Plancherel measure** on $\mathbb{Y}$ assigns to λ the probability

$$\frac{(f^\lambda)^2}{n!}, \quad \text{where } f^\lambda \text{ is the number of standard Young tableaux of shape } \lambda.$$

Let $d^\lambda(q) := \#\texttt{INC}(\lambda, q)$ and $e^\lambda(n) = \#\texttt{SsetT}(\lambda, n)$. (Note $f^\lambda = e^\lambda(|\lambda|)$.)

Definition 1.1. The **Plancherel–Hecke probability measure** $\mu_{n,q}$ on $\mathbb{Y}$ is defined by

$$\mu_{n,q}(\lambda) := \frac{1}{q^n} d^\lambda(q) e^\lambda(n).$$

Proposition 1.2. *The Plancherel–Hecke measure is well defined as a probability distribution in* $\mathbb{Y}$; *i.e., the following identity holds*

$$q^n = \sum_\lambda d^\lambda(q) e^\lambda(n), \tag{1}$$

where the sum is over all $\lambda \in \mathbb{Y}$.

There is an exact polynomial-time sampling algorithm

$$\texttt{Heckeshape} : W_{n,q} \to \mathbb{Y},$$

terminating in $O(nq)$ *operations, that induces* $\mu_{n,q}$ *from the uniform distribution on* $W_{n,q}$.

Note that (1) is also an identity when we restrict the sum to be over

$$|\lambda| \leqslant n \quad \text{and} \quad \lambda \subseteq (q, q-1, q-2, \ldots, 3, 2, 1), \tag{2}$$

since otherwise $d^\lambda(q)e^\lambda(n) = 0 (= Prob(\lambda_{n,q} = \lambda))$.

The core technical result of this paper is a generalization of the aforementioned theorem of Schensted [20]:

Theorem 1.3. $\texttt{Heckeshape}$ *simultaneously and symmetrically encodes* $\texttt{LIS}(w)$ *and* $\texttt{LDS}(w)$ *as the size of the first row and column of* $\texttt{Heckeshape}(w)$, *respectively.*

Here $\mathtt{Heckeshape}(w)$ is the Young diagram associated to w under `Hecke`. Theorem 1.3 is obtained by establishing another new result, connecting `Heckeshape` to the "K-infusion" operation defined in [24]. This latter result is an analogue of the classical fact that connects the Robinson–Schensted correspondence to the (ordinary) *jeu de taquin* rectification procedure.

We prove Proposition 1.2 in Section 2, after recalling the `Hecke` algorithm of Buch, Kresch, Shimozono, Tamvakis and Yong [7] (originally constructed to study degeneracy loci of vector bundles). The proof of Theorem 1.3 is given in Section 4.

Example 1.4. We illustrate the identity (1) for $n = 4$ and $q = 3$. There are nine partitions λ satisfying (2). These are

$$(1),\ (2),\ (1,1),\ (2,1),\ (3),\ (1,1,1),\ (3,1),\ (2,1,1),\ (2,2).$$

Then (1) reads

$$81 = 3^4 = 3\cdot 1 + 3\cdot 3 + 3\cdot 3 + 5\cdot 8 + 1\cdot 3 + 1\cdot 3 + 2\cdot 3 + 2\cdot 3 + 1\cdot 2,$$

where the products on the right-hand side of the equality are listed in order corresponding to the above partitions. Thus, the "typical shape" is (2, 1), possessing nearly 50% of the distribution. The remainder of the above results will be illustrated in Section 2, after we define `Heckeshape`.

Theorem 1.3 has some immediate consequences, familiar from the permutation case.

Corollary 1.5. *Under the uniform measure on $W_{n,q}$, we have*

$$E(\mathtt{LIS}) = \frac{1}{q^n}\sum_{\lambda\in\mathbb{Y}} \lambda_1 d^\lambda(q) e^\lambda(n). \tag{3}$$

In addition,

$$Prob(\mathtt{LIS} = \ell) = \frac{1}{q^n}\sum_{\lambda\in\mathbb{Y},\,\lambda_1=\ell} d^\lambda(q) e^\lambda(n).$$

Two other consequences will be given in Section 3. One gives a "Coxeter-theoretic" generalization of the widely known Erdős–Szekeres theorem [10]. Another expands upon the discussion of patience sorting given in [1].

1.3. *Remarks on Proposition 1.2 and Theorem 1.3*

In our experiments, `Heckeshape` was reasonably efficient as a sampling algorithm.[1] For example, when $n \leqslant 10{,}000$, sampling one (i.e., random generation of one) Young diagram takes on the order of seconds to minutes on current technology. For larger n, we could sample one Young diagram when $n = 50{,}000$ in several hours on the same technology. A sample when $n = 100{,}000$ took about one and a half days. The memory demands were modest. In view of the apparent "concentration" suggested below, one sample was enough to be of interest for our purposes, when n is large.

There are classical antecedents of Theorem 1.3. As stated earlier, Schensted [20] proved the analogous conclusion about the shape coming from the Robinson–Schensted correspondence for a permutation w. In contrast, Knuth [16] proved that the first row of the Robinson–Schensted–Knuth algorithm (`RSK`) encodes the length of the longest weakly increasing subsequence (`LwIS`) of a word w.

[1] Software available at the authors' websites.

What is perhaps less well known is that RSK also encodes LDS(w) as the length of the first column of the shape it associates to w. However, unlike Heckeshape, it is asymmetric: LIS(w) is not encoded by the length of the first row (as LIS(w) $\neq$ LwIS(w) in general). Thus, the symmetry of Hecke(shape) makes it natural to analyze, as it seems desirable to simultaneously capture the statistics of LIS and LDS.

However, we do not have handy formulas for $Prob(\lambda)$, $d^{\lambda}(q)$ or $e^{\lambda}(n)$, such as the *hook-length formula* for f^{λ}. (In the permutation case, the hook-length formula plays a crucial role, see [26,18].) In small examples large prime factors appear, showing that such a formula is unlikely. For instance, $d^{(4,2,1)}(7) = 1337 = 7 \cdot 191$ and $e^{(4,2,1)}(8) = 452 = 2^2 \cdot 113$. This issue is closely related to the open question of finding "good" determinantal expressions for *Grothendieck polynomials* [17]. That being said, special cases exhibit connections to work of Stanley [21] on polygonal dissections, and of [11] on generalized Littlewood–Richardson rules (further discussion may appear elsewhere). Thus, the enumerative combinatorics of these numbers might of interest in their own right.

It is not difficult to give recursions to calculate $d^{\lambda}(q)$ and $e^{\lambda}(n)$ that are useful in moderately large cases. Are there efficient (possibly randomized or approximate) counting algorithms?

Objectively, the lack of simple formulas to compute $Prob(\lambda)$ makes it trickier to apply standard approaches directly; this is an admitted defect of our setting. Nevertheless, we believe the framework of problems described here is tractable. In addition to the results below, in Appendix A one gains useful and nontrivial information about the Plancherel–Hecke measure by exploiting the related work of [3]. In this way, the techniques of [18,26] can be applied to the present context.

1.4. Analysis of $\mu_{n,q}$ and the limit typical shape

We organize our analysis by first setting

$$q = H(n) \in \Theta\left(n^{\alpha}\right), \quad \text{where } 0 < \alpha \leqslant 1. \tag{4}$$

Here we are using the standard "Big Theta notation" from computational complexity in writing "$H(n) \in \Theta(n^{\alpha})$". This means that there exist fixed constants k_1, k_2, n_0 such that $k_1 n^{\alpha} < H(n) < k_2 n^{\alpha}$ for $n > n_0$. We are then interested in the behavior of $\mu_{n,q}$ as $n \to \infty$ (and thus $q = H(n) \to \infty$). (The case $\alpha = 0$ is trivial.) As is explained below, we conjecture that there is a **critical value** of α, denoted $\alpha_{\text{critical}} := \frac{1}{2}$: the behavior of $\mu_{n,q}$ is qualitatively different in the intervals $\alpha \in (\alpha_{\text{critical}}, 1]$ and $\alpha \in (0, \alpha_{\text{critical}})$. At $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$, further refinement of the analysis is needed, as we transition from one state to the other.

Informally stated, in the permutation case, to study the Plancherel measure, it is useful to consider the most likely, or "typical" shape. There, three facts are true. First, in the large limit (and after rescaling), a well-defined typical shape exists. Second, the expectation of the LIS and LDS of a large random permutation is encoded respectively in the length of the first row and first column of the limit shape. Third, the Plancherel measure is concentrated near the typical shape. For a detailed explanation see [1] and the references therein.

We conjecture that analogues of all three of the aforementioned features also hold for the Plancherel–Hecke measure.

To be more precise, let the **typical shape** $\Lambda_{n,q}$ be the shape λ (contained in $(q, q-1, \ldots, 2, 1)$) maximizing $Prob(\lambda)$. This Young diagram $\Lambda_{n,q}$ can be drawn in the plane in "French notation" using unit squares (so the first column is of length $(\Lambda_{n,q})_1$, etc.) Let $f_{n,q} : [0, \infty) \to \mathbb{R}_{\geqslant 0}$ be the function whose graph consists of the horizontal segments of this boundary. Finally, rescale by

$$\hat{f}_{n,q}(x) := \frac{f_{n,q}(cx)}{c}, \quad \text{where } c = \min(q, 2\sqrt{n}).$$

Conjecture 1.6.

(I) *For any* $0 \leqslant \alpha \leqslant 1$, *there is a unique continuous function*

$$\Lambda \in C\big([0,\infty) \to \mathbb{R}_{\geqslant 0}\big)$$

such that for any $\epsilon > 0$,

$$Prob\Big(\sup_{x\in\mathbb{R}_{\geqslant 0}} \big|\hat{f}_{n,q} - \Lambda\big| > \epsilon\Big) \to 0$$

as $n \to \infty$. *We call this* Λ *the* **limit typical shape**.

(II) *A "phase transition" occurs at* $\alpha_{\text{critical}} = \frac{1}{2}$:
For $0 < \alpha < \alpha_{\text{critical}} = \frac{1}{2}$, Λ *is the line*

$$y = 1 - x, \quad \text{for } 0 \leqslant x \leqslant 1. \tag{5}$$

For $\alpha_{\text{critical}} = \frac{1}{2} < \alpha \leqslant 1$, Λ *is the* **Plancherel curve**, *which is parametrically given by*

$$x = y + \cos\theta, \qquad y = \frac{1}{\pi}(\sin\theta - \theta\cos\theta), \quad \text{for } 0 \leqslant \theta \leqslant \pi. \tag{6}$$

(*The curves defined by* (5) *and* (6) *are declared to be identically* 0 *for* $1 \leqslant x < \infty$.)

(III) *For* $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$: *there is a constant* $C > 0$ *such that if*

$$q = kn^{\frac{1}{2}} + o\big(n^{\frac{1}{2}}\big),$$

then if $k < C$, Λ *is the function given by* (5). *Otherwise,* Λ *is a function given by a deformation of* (6) *which is symmetric across the line* $y = x$. *In either case, the* x *and* y *intercepts of the function* Λ *are at*

$$0 \leqslant \beta(k) \leqslant 1$$

where

$$E(\texttt{LIS}) \approx \beta(k)2\sqrt{n},$$

i.e., $\lim_{n\to\infty} E(\texttt{LIS})/2\sqrt{n} = \beta(k)$, *and explicitly,*

$$\beta(k) = \begin{cases} \frac{k}{2} & \text{if } 0 < k \leqslant 1, \\ \frac{2-k^{-1}}{2} & \text{if } k > 1. \end{cases} \tag{7}$$

We have reasonable support for the cases (I) and (II) of Conjecture 1.6. *Heuristically*, part (II) of the conjecture says that when α is large, and thus q is "close" to n, a random word is "close to" being a random permutation. For a permutation, Schensted and Hecke behave the same. Hence the Plancherel and Plancherel–Hecke measures ought to be maximized on the same shape. When α is small, the limit typical shape is a rescaling of the "staircase shape" which plays a distinguished role in the Edelman–Greene algorithm [9] and the Hecke algorithm; see Theorem 1.8 and its proof.

Conjecture 1.6(III) is more speculative, since we did not have as much computational evidence for the shape of Λ. There appears to be a continuous "flattening" of the Plancherel curve to a line as we

Table 1
Estimates of $\beta(k)$ for the $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$ case.

k	0.5	1	2	4	10
$\beta(k)$ estimate	0.25	0.50	0.74	0.86	0.94

tune k from ∞ to 0. Our data was insufficient to rule out the possibility that Λ is simply a rescaling of the Plancherel curve by a factor of $\beta(k)$.

Problem 1.7. Explicitly describe the deformation of Λ when $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$, as k varies.

Our best estimate is that $\frac{1}{2} \leqslant C \leqslant 1$ (probably just $C = 1$). However, the values of $\beta(k)$ for k relatively small can be experimentally estimated. Table 1 below was based on Monte Carlo estimates of $E(\mathtt{LIS})$ for $n = 50{,}000$, $100{,}000$, $200{,}000$ and $300{,}000$ and the estimates were stable throughout this range. They closely agree with the conjecture for $\beta(k)$ given above.

Notice that since $Prob(\lambda) = Prob(\lambda')$ where λ' is the conjugate shape of λ we know that if Λ exists and is unique, then Λ is symmetric. This is consistent with the limit curves we predict.

We prove in Section 5 that:

Theorem 1.8. *Conjecture 1.6 is true for $0 \leqslant \alpha < \frac{1}{3}$. More precisely, in this range, a random shape λ under $\mu_{n,q}$ satisfies $\lambda_i = q - i + 1$ almost surely, as $n, q \to \infty$.*

The proof of this theorem depends on the analysis of a certain random walk on the symmetric group. Our analysis is not sharp enough to extend to the range $\frac{1}{3} \leqslant \alpha \leqslant \alpha_{\text{critical}} = \frac{1}{2}$, although a refinement might be possible.

Empirically, one finds that the first row and column of $\Lambda_{n,q}$ are approximations of $E(\mathtt{LIS})$ and $E(\mathtt{LDS})$ that improve as $n \to \infty$. Therefore, it makes sense to study the asymptotics of $E(\mathtt{LIS})$ and $E(\mathtt{LDS})$ as a means to understand the characteristics of $\Lambda_{n,q}$. From this point of view, the following result supports the phase transition phenomena asserted in Conjecture 1.6.

Theorem 1.9 *(With O. Zeitouni). If $0 \leqslant \alpha < \alpha_{\text{critical}} = \frac{1}{2}$ then $\lim_{n\to\infty} E(\mathtt{LIS}) = q$, whereas if $\alpha_{\text{critical}} = \frac{1}{2} < \alpha \leqslant 1$ then $E(\mathtt{LIS}) \approx 2\sqrt{n}$. The same statements hold when $E(\mathtt{LDS})$ replaces $E(\mathtt{LIS})$.*

The proof for $\alpha < \alpha_{\text{critical}} = \frac{1}{2}$ is a variation on the approach we use to prove Theorem 1.8. We also conjectured the answer for $\alpha > \alpha_{\text{critical}} = \frac{1}{2}$; after showing O. Zeitouni our guess during an early stage in the project, he communicated to us a proof, and kindly allowed us to reproduce his argument here.

In private communication, E. Rains offered a proof that $E(\mathtt{LIS}) = q$ in the $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$ and $k \leqslant 1$ case. Afterward, in a supplementary text to this paper (now included as an Appendix A), O. Zeitouni and the second author present a simple proof that $E(\mathtt{LIS}) \approx \beta(k)2\sqrt{n}$, for all k, and where here $\beta(k)$ is defined by (7), in the $\alpha = \alpha_{\text{critical}} = \frac{1}{2}$ case. This closes the gap in Theorem 1.9. The proof builds on work of [3] (see further discussion in Section 1.5). These results further support the belief that $C = 1$ in Conjecture 1.6(III).

In addition, we have the following conjecture about the fluctuation of LIS and LDS.

Conjecture 1.10. *Let $\sigma(\mathtt{LIS})$ denote the standard deviation of LIS. For $0 < \alpha < \alpha_{\text{critical}} = \frac{1}{2}$ then*

$$\lim_{n\to\infty} \sigma(\mathtt{LIS}) = 0,$$

whereas if $\alpha_{\text{critical}} = \frac{1}{2} < \alpha \leqslant 1$ then

$$\lim_{n\to\infty} \sigma(\mathtt{LIS}) = O\left(n^{\frac{1}{6}}\right).$$

The same statements hold for LDS.

Table 2
$n = 50{,}000$ with 1,000 Monte Carlo trials.

Estimate \ α	0.45	0.50	0.55	0.60	0.75	1.00
$E(\texttt{LIS})$	130.00	222.50	311.06	368.38	422.48	436.36
σ	0.00	0.63	2.86	4.01	5.07	5.09

Table 3
$n = 100{,}000$ with 500 Monte Carlo trials.

Estimate \ α	0.45	0.50	0.55	0.60	0.75	1.00
$E(\texttt{LIS})$	177.00	315.43	448.2	523.63	603.78	619.64
σ	0.00	0.67	3.52	4.62	5.90	6.29

Note that Theorem 1.8 implies Conjecture 1.10 holds for $0 \leqslant \alpha < \frac{1}{3}$. Tables 2 and 3 give numerical evidence for Conjecture 1.10 and are consistent with Theorem 1.9 as the estimated values for σ are on the order of $n^{\frac{1}{6}}$.

The bulk of $\mu_{n,q}$ appears "concentrated" near $\Lambda_{n,q}$, i.e., the probability of sampling a random shape differing, in the sup-norm, from Λ after rescaling, by some fixed $\epsilon > 0$, goes to 0 as $n, q \to \infty$. See Fig. 2: already at $n = 5{,}000$ we see that two random samples are visibly "close" to one another, and are similar in shape to the third curve which is an approximation of the Plancherel curve. By $n = 100{,}000$ the curves appear undeniably to be rescalings of one another, with a rescaling factor of about 0.95 in our experiments. Naturally, as α gets larger, the empirical convergence of the curves occurs faster.

1.5. Further comparisons with the literature

As mentioned earlier, the limit distribution of LIS on permutations, and that of LwIS on words is well understood.

The study of LIS on $W_{n,q}$ was considered, e.g., in [25]. In addition, the study of the distribution of LIS, in the critical case $\alpha_{\text{critical}} = \frac{1}{2}$ is implicit in [3,5]. In [3], an alternative measure on Young diagrams is studied: **Schur–Weyl duality** implies that one has the decomposition

$$\left(\mathbb{C}^q\right)^{\otimes n} \cong \bigoplus_{\lambda} S^{\lambda} \otimes V_{\lambda},$$

where here S^{λ} is the S_n irreducible *Specht module* and V_{λ} is the $GL_q(\mathbb{C})$ irreducible *Schur module*. Now taking dimensions one defines a probability measure that assigns to λ the likelihood $(\dim S^{\lambda} \cdot \dim V_{\lambda})/q^n$. Biane explicitly determines the rescaled limit typical shape in this context. Combinatorially, Biane's measure arises from the RSK algorithm. Since we know that $\texttt{RSK}(w)$ encodes the $\texttt{LIS}(w)$ in the first column (by reading w backwards), one expects, by analogy with [18,26] that a certain rescaling of the first column of Biane's limit shape is the $\beta(k)$ of Conjecture 1.6. However, to justify this conclusion rigorously one needs more work. Further, the fluctuations around Biane's curve have been studied, in the $k = 1$ case, by Borodin and Olshanski [5].

Hecke was originally developed in [7] as a generalization of the *Edelman–Greene correspondence* which bijects Coxeter reduced words in the symmetric group to pairs of tableaux [9]. Our proof of Theorem 1.3 implies that this algorithm encodes the LIS of such words, although the study of LIS of reduced words appears unmotivated. On the other hand, the Coxeter-theoretic viewpoint on words will be useful in our analysis of LIS, LDS and Λ.

1.6. Summary and organization

In Section 2 we recall the Hecke algorithm and give an additional example of the results of Section 1.2. We then prove Proposition 1.2. In Section 3, we include two consequences of Theorem 1.3.

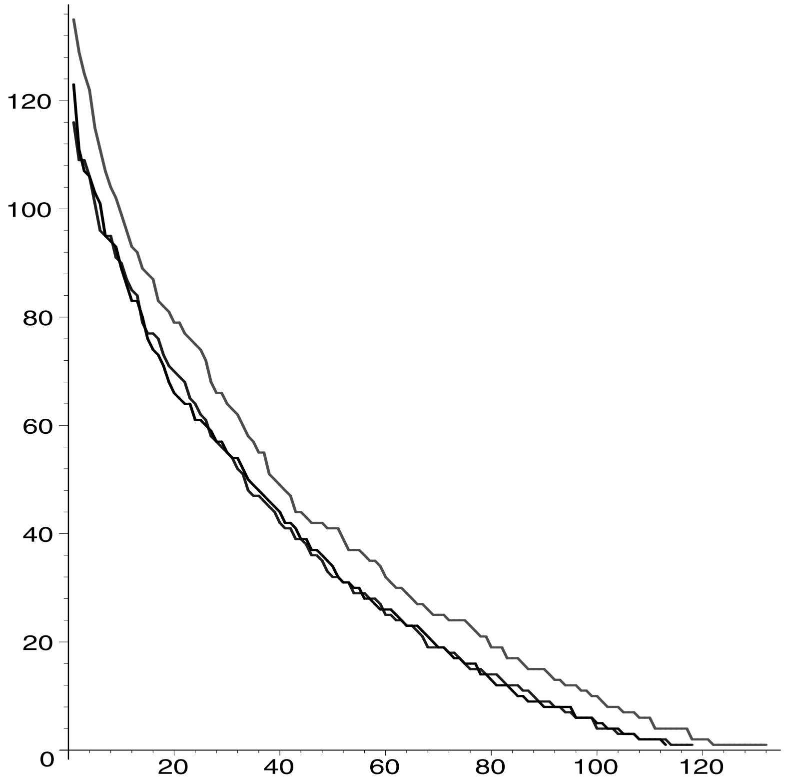

Fig. 2. Two samples at $n = 5000$, $q = \lfloor 5000^{\frac{2}{3}} \rfloor$ compared with an empirical approximation of the Plancherel curve; conjecturally as $n \to \infty$, the sample curves converge to one another.

We split our remaining proofs according to the main flavor of technique used: in Section 4 we explain the increasing tableau theory we need from [24] and prove Theorem 1.3. In Section 5, we utilize probabilistic-combinatorial techniques, combined with our main results, to prove Theorems 1.8 and 1.9.

2. The `Hecke` algorithm

2.1. The 0-Hecke monoid

We need to recall some notions used in [7]. The **0-Hecke monoid** $\mathcal{H}_{0,q}$ is the quotient of the free monoid of all finite words in the alphabet $\{1, 2, \ldots, q\}$ by the relations

$$ii \equiv i \quad \text{for all } i, \tag{8}$$

$$iji \equiv jij \quad \text{for all } i, j, \tag{9}$$

$$ij \equiv ji \quad \text{for } |i - j| \geqslant 2. \tag{10}$$

There is a bijection between $\mathcal{H}_{0,q}$ and the symmetric group S_{q+1}. Given any word $a \in \mathcal{H}_{0,q}$ there is a unique permutation $\pi \in S_{q+1}$ such that $a \equiv b$ for any reduced word b of π; see, e.g., the textbook [4] for basic Coxeter theory for the symmetric group. In this case, we write $W(a) = \pi$ and say that a is a **Hecke word** for π. Indeed, the reduced words for π are precisely the Hecke words for π that are of the minimum length $\ell(\pi)$, the **Coxeter length** of π (after identifying the label i with the simple reflection $s_i = (i\ i+1)$). Given an additional permutation ρ with Hecke word b, the **Hecke product** of π and ρ is defined as the permutation $\pi \cdot \rho = W(ab)$.

The (row reading) word of a tableau T, denoted $\mathtt{word}(T)$, is obtained by reading the rows of the tableau from left to right, starting with the bottom row, followed by the row above it, etc. We also define $W(T) := W(\mathtt{word}(T))$. So for example, if T is the increasing tableau of Fig. 1, then $\mathtt{word}(T) =$ 5 3 4 1 3 4 5.

The (general) `Hecke` algorithm defined in [7] identifies pairs (w, i) of words

$$w = w_1 w_2 \cdots w_n, \qquad i = i_1 i_2 \cdots i_n,$$

where w is a **Hecke word** and i satisfies

$$i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \quad \text{and} \quad i_j < i_{j+1} \quad \text{whenever } w_j \leqslant w_{j+1},$$

with pairs of tableaux (P, Q) of the same shape, where P is an increasing tableau such that $\mathtt{word}(P) \equiv w$ and the **content** (i.e., multiset of labels) of Q matches the content of i. We refer to the P-tableau as the **insertion tableau** and the Q-tableau as the **recording tableau**. (We point out that the "column" convention in [7] differs slightly from the "row" one used here.)

2.2. *Description of* `Hecke` *and* `Heckeshape`

The following description of (general) `Hecke` was originally given in [7]:

Description of (*general*) `Hecke`: In this algorithm, one iteratively inserts an integer x into an increasing tableau T. We denote each such insertion by $T \leftarrow x$. The output is a triple (U, c, α) where U is a modification of T (possibly $T = U$), c is a corner of U and $\alpha \in \{0, 1\}$ is a parameter. Initially, we attempt to insert x into the first row of T, and an output integer is possibly created which is inserted into the next row and so on, until no output integer is created. We refer to this final insertion as the **terminating step**, and the previous insertions as **bumping steps**.

Suppose R is a row that we are attempting to insert x into. If x is larger than or equal to all the entries of R, then no output integer is generated and the algorithm terminates: if adjoining x to the end of R results in an increasing tableau U, then set $\alpha = 1$ and c to be the new corner added. Otherwise end with the present U, without modification; $\alpha = 0$ and c is the corner that is at the end of the column containing the rightmost box of R. On the other hand, if R contains boxes strictly larger than x, let y be the smallest such box. If replacing y with x results in an increasing tableau, then do so. In either case, y is the output integer to be inserted into the next row.

Inserting a word w using this algorithm terminates with an increasing tableau

$$P = \Big(\Big(\big((\emptyset \leftarrow w_1) \leftarrow w_2\big) \leftarrow \cdots\Big) \leftarrow w_n\Big).$$

The Q tableau is obtained by placing each i_j in the c-corner resulting from the insertion of w_j.

We also have the following reverse insertion algorithm $\mathtt{Hecke}^{-1}$.

Description of (*general*) $\mathtt{Hecke}^{-1}$: Let Z be an increasing tableau, c a corner of Z, and $\alpha \in \{0, 1\}$. Reverse insertion applied to the triple (Z, c, α) produces a pair (Y, x) of an increasing tableau Y and a positive integer x as follows. Let y be the integer in the cell c of Z. If $\alpha = 1$, remove y. In any case, reverse insert y into the row above the corner c.

Whenever a value y is reverse inserted into a row R, let x be the largest entry of R such that $x < y$. If replacing y with x results in an increasing tableau, then this is done. In any case, the integer

x is passed up. If R is not the top row, this means that x is reverse inserted into the row above of R; otherwise x becomes the final output value, along with the modified tableau.

We now complete the description of (general) `Hecke`. Locate the bottom-most corner with the largest label, in the Q tableau, and remove the label. If it was the only entry in its corner, remove the corner, set $\alpha = 1$. Otherwise set $\alpha = 0$. Set c to be this corner. Then reverse insert (P, c, α). Repeat until all the entries of Q (and P) have been removed.

`Hecke` is a generalization of the Robinson–Schensted correspondence in the sense that it agrees with that correspondence whenever w is a permutation in S_n and $i_j = j$ for all j. In that case the P and Q tableaux are both standard Young tableaux.

In this paper, we are only concerned with the case $i_j = j$. Therefore, we also set

$$\texttt{Hecke}(w) := \texttt{Hecke}(w, 123\cdots n).$$

Also, we define

$$\texttt{Heckeshape} : W_{n,q} \to \mathbb{Y}$$

by setting `Heckeshape`(w) to be the common shape of P and Q under `Hecke`(w). (An alternative description of this map is given in Theorem 4.2 in Section 4.)

Example 2.1. Let $w = 5\ 4\ 1\ 3\ 4\ 2\ 5\ 1\ 2\ 1\ 4\ 2\ 4 \in W_{13,5}$. Then the reader can check that `Hecke` produces the following steps:

$$\begin{array}{l}5\end{array}, \begin{array}{l}1\end{array} \mapsto \begin{array}{l}4\\5\end{array}, \begin{array}{l}1\\2\end{array} \mapsto \begin{array}{l}1\\4\\5\end{array}, \begin{array}{l}1\\2\\3\end{array} \mapsto \begin{array}{ll}1&3\\4\\5\end{array}, \begin{array}{ll}1&4\\2\\3\end{array} \mapsto \begin{array}{lll}1&3&4\\4\\5\end{array}, \begin{array}{lll}1&4&5\\2\\3\end{array} \mapsto \begin{array}{lll}1&2&4\\3\\4\\5\end{array}, \begin{array}{lll}1&4&5\\2\\3\\6\end{array}$$

$$\mapsto \begin{array}{llll}1&2&4&5\\3\\4\\5\end{array}, \begin{array}{llll}1&4&5&7\\2\\3\\6\end{array} \mapsto \begin{array}{llll}1&2&4&5\\2\\3\\4\\5\end{array}, \begin{array}{llll}1&4&5&7\\2\\3\\6\\8\end{array} \mapsto \begin{array}{llll}1&2&4&5\\2&4\\3\\4\\5\end{array}, \begin{array}{llll}1&4&5&7\\2&9\\3\\6\\8\end{array}$$

$$\mapsto \begin{array}{llll}1&2&4&5\\2&4\\3\\4\\5\end{array}, \begin{array}{llll}1&4&5&7\\2&9\\3\\6\\8,10\end{array} \mapsto \cdots$$

$$\mapsto \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 2 & 4 & 5 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 5 & 7 \\ \hline 2 & 9 & 11,13 \\ \cline{1-3} 3 & 12 \\ \cline{1-2} 6 \\ \cline{1-1} 8,10 \\ \cline{1-1}\end{array}.$$

Here $\texttt{Heckeshape}(w) = (4,3,2,1,1)$ and indeed the length of the first row of this shape equals $\texttt{LIS}(w) = 4$, whereas the length of the first column equals $\texttt{LDS}(w) = 5$.

2.3. Proof of Proposition 1.2

The claim that $\mu_{n,q}$ is a probability distribution follows if $\texttt{Hecke}$ extends to provide a bijection between:

$$W_{n,q} \quad \text{and} \quad \Gamma_{n,q} := \bigcup_{\lambda} \texttt{INC}(\lambda, q) \times \texttt{SsetVT}(\lambda, n),$$

where λ satisfies (2).

Associate to each word $w \in W_{n,q}$ the pair $(w, 123\cdots n)$. Clearly $\texttt{Hecke}$ injectively maps these pairs into $\Gamma_{n,q}$.

To prove surjectivity, let $(P, Q) \in \Gamma_{n,q}$. Then under $\texttt{Hecke}^{-1}$, (P, Q) corresponds to some pair (w, i). Now $i = 123\cdots n$ since that is the only possible sequence that can arise from a standard tableau Q. Also, since $W(w) = W(\texttt{word}(P))$, w must use some subset of $\{1, 2, \ldots, q\}$. Thus $w \in W_{n,q}$. Hence $W_{n,q} \twoheadrightarrow \Gamma_{n,q}$. The claim (2) is then clear from the properties of $\texttt{Hecke}$.

Finally, from the above discussion it is immediate that $\texttt{Heckeshape}$ is a sampling algorithm for $\mu_{n,q}$. The bottleneck of the algorithm is the insertion process (a random uniform word $w \in W_{n,q}$ can be generated in $O(n \log(q))$ time). By (2) we know that each of the n insertions demand at most $2q$ operations. Hence $O(nq)$ operations are needed. This completes the proof of Proposition 1.2.

Remark 2.2. In a preprint version of this paper, we had a bound of $O(nq^2)$ on the complexity. We thank the anonymous referee for a simplification that lowers the bound.

3. Some further consequences of Theorem 1.3

3.1. A generalization of the Erdős–Szekeres theorem

The following classic result is due to Erdős and Szekeres [10]:

Theorem 3.1. *Let $a, b \geqslant 1$. If $w \in S_{ab+1}$ then $\texttt{LIS}(w) > a$ or $\texttt{LDS}(w) > b$.*

It is known that this result can be readily deduced from Schensted's results, see, e.g., [23, Section 2]. Theorem 1.3 similarly leads to an extension of Theorem 3.1 that relates $\texttt{LIS}$ and $\texttt{LDS}$ to Coxeter length.

Proposition 3.2. *Let $w \in W_{n,q}$. Suppose $1 \leqslant a, b < q$ and*

$$\ell\bigl(W(w)\bigr) > \sum_{i=1}^{a} \min(b, q-i), \quad \textit{or equivalently} \quad \ell\bigl(W(w)\bigr) > \sum_{j=1}^{b} \min(a, q-j), \tag{11}$$

then $\mathtt{LIS}(w) > a$ *or* $\mathtt{LDS}(w) > b$; *recall $W(w)$ is the permutation identified with w.*

Proof. If $\mathtt{LIS}(w) \leqslant a$ and $\mathtt{LDS}(w) \leqslant b$ then by Proposition 1.2 and Theorem 1.3, we have

$$\mathtt{Heckeshape}(w) \subseteq (a \times b) \cap (q, q-1, \ldots, 3, 2, 1).$$

Thus

$$\bigl|\mathtt{Heckeshape}(w)\bigr| \leqslant \sum_{i=1}^{a} \min(b, q-i) = \sum_{j=1}^{b} \min(a, q-j).$$

Since $\ell(W(w)) \leqslant |\mathtt{Heckeshape}(w)|$, the result then follows. □

Example 3.3. If $q = 4$ and $a = b = 3$ then if $\ell(W(w)) > 8$ then $\mathtt{LIS}(w) > 3$ or $\mathtt{LDS}(w) > 3$. This inequality is tight in the sense that the bound 8 cannot be reduced: consider $w = 2\ 1\ 3\ 4\ 2\ 3\ 1\ 2$. This is already a reduced word of Coxeter length 8, viewed as an element of S_5, and $\mathtt{LIS}(w) = \mathtt{LDS}(w) = 3$.

Proposition 3.2 generalizes Theorem 3.1 because if $w \in S_{ab+1}$ is viewed as a Hecke word, we have $\ell(w) = ab + 1$ (any word where all letters are distinct is automatically reduced). Then set $q = ab + 1$ and thus (11) is satisfied.

3.2. Patience sorting for decks with repeated values

In [1], the Schensted correspondence was connected to the one-person (solitaire) card game **patience sorting**. We include a generalization of this connection, which in particular is a refinement of the $\mathtt{LIS}$ claim of Theorem 1.3.

In this game, a deck of cards labeled $1, 2, \ldots, n$ is shuffled and the cards are turned up one at a time and dealt into piles on the table: a lower card may be placed on top of a higher card, or put into a new pile to the right of the existing piles. The goal of the game is to finish with as few piles as possible.

For example, if $n = 10$ and the deck is shuffled in the order

$$8\ 2\ 6\ 3\ 4\ 1\ 7\ 10\ 9$$

then the top card 8 is dealt onto the table. The 2 can either be placed to the right of the 8 or on top of it – suppose we chose the latter scenario. Next the 6 must be placed to the right of the pile containing the 2 and 8, starting a new pile. At this stage, we have $\begin{matrix} 2 & \\ 8 & 6 \end{matrix}$.

The **greedy strategy** is to always place the new card in the leftmost pile possible. If we complete the game using this strategy, we would obtain, successively:

$$\begin{matrix} 2 & 3 \\ 8 & 6 \end{matrix} \mapsto \begin{matrix} 2 & 3 & \\ 8 & 6 & 4 \end{matrix} \mapsto \begin{matrix} 1 & & \\ 2 & 3 & \\ 8 & 6 & 4 \end{matrix} \mapsto \begin{matrix} 1 & & & \\ 2 & 3 & & \\ 8 & 6 & 4 & 7 \end{matrix} \mapsto \begin{matrix} 1 & & & & \\ 2 & 3 & & & \\ 8 & 6 & 4 & 7 & 10 \end{matrix} \mapsto \begin{matrix} 1 & & & & \\ 2 & 3 & & & 9 \\ 8 & 6 & 4 & 7 & 10 \end{matrix}.$$

It is easy to prove that the top cards increase from left to right throughout the game; Mallows [19] and later independently Hammersley [13, p. 362] observed that the number of piles at the end equals $\mathtt{LIS}(w)$, where $w \in S_n$ is the permutation defining the shuffled deck. Finally, Aldous–Diaconis note that the first row of the insertion tableau under Robinson–Schensted agrees with the top cards.

Aldous and Diaconis [1, Section 2.4] consider two variants of patience sorting where the deck has repeated entries, i.e., where all cards of the same rank (e.g., all Jacks) are equal. The two rules they consider are "ties forbidden" and "ties allowed", depending on whether or not a Jack can be placed on top of another Jack. They provide an analysis of the former case, relating it to the Robinson–Schensted–Knuth correspondence.

For example, if the shuffled deck is given by $w = 2\ 1\ 4\ 1\ 3\ 5\ 3\ 2\ 5\ 1\ 4\ 2 \in W_{12,5}$, then the result of playing patience (using the greedy strategy) with ties forbidden and allowed, respectively, are

$$\begin{array}{ccccc} & & & & \\ & 1 & 2 & & \\ 1 & 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 5 & 5 \\ & & & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & 2 & \\ 1 & 2 & \\ 1 & 3 & 4 \\ 1 & 3 & 5 \\ 2 & 4 & 5 \end{array}.$$

Proposition 3.4. *Assume patience sorting is played with ties allowed, on a deck of n cards with q distinct types of cards (viewed as a word $w \in W_{n,q}$). Then*

(I) *The top cards of each pile at the termination of the game, using the greedy strategy, as read from left to right, agree with the top row of the insertion tableau of* $\mathtt{Hecke}(w)$.
(II) *The greedy strategy is optimal (i.e., it minimizes the number of piles);* $\mathtt{LIS}(w)$ *piles are created.*

Proof. The proof of (I) is an easy induction, comparing the description of $\mathtt{Hecke}$ with the "tied allowed" rules of patience sorting.

For (II), by (I) and Theorem 1.3 we know the "greedy strategy" terminates with $\mathtt{LIS}(w)$ piles. The proof of optimality is the same *mutatis mutandis* as the one for the original variant, see [1, Lemma 1]. □

Briefly, probabilistic and statistical analysis on the "tied allowed" case of patience sorting is possible, in analogy to the work of [1]. Below we have tabulated the results of a Monte Carlo simulation with 100,000 trials on a standard 52-card deck. Typically, the number of piles is between 8 and 11. The average number of piles is 9.2 which, naturally, is less than the average number of piles when the deck is totally ordered, which is 11.6 as reported by Aldous–Diaconis. So a deck ordering is "lucky" if the number of piles is less than 7, which occurs only about 3% of the time (see Table 4).

Table 4
Monte Carlo simulation for standard 52-card deck with 100,000 trials. The average number of piles is 9.2.

Number of piles	6	7	8	9	10	11	12	13
Frequency	82	2993	20 336	39 039	27 843	8489	1166	52

Fig. 3 shows that there is a definite shape describing the mean pile sizes as α varies. Questions about the structure of this shape may be interpreted as enriched questions related to $\mathtt{LIS}$. One can analyze such questions using the dichotomy of Section 1.4; we do not pursue this here.

4. Increasing tableau theory and the proof of Theorem 1.3

Now we show that the size of the first row of $\lambda = \mathtt{Heckeshape}(w)$ computes $\mathtt{LIS}(w)$. Let $r(w,t)$ be the largest index such that the longest strictly increasing subsequence ending at $w_{r(w,t)}$ has length t. For example, if $w = 2\ 3\ 4\ 1\ 5\ 2$ then $r(w,1) = 4$, $r(w,2) = 6$ and $r(w,3) = 3$. We now

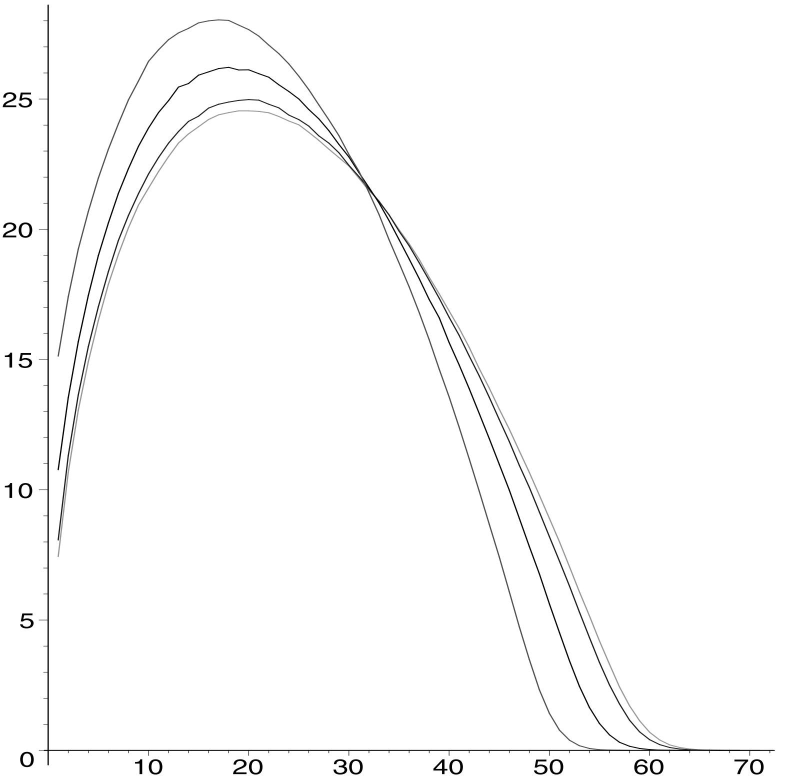

Fig. 3. Simulation of mean pile sizes for $n = 1000$ and $q = 100, 200, 600, 1000$ with 10^4 Monte Carlo trials. The x-axis indicates the position of the pile, counting from the left.

in fact prove the following claim, which is stronger than the LIS assertion of Theorem 1.3 (cf. [22, Proposition 7.23.10]):

Proposition 4.1. *Suppose $w \in W_{n,q}$, $s = \mathtt{LIS}(w)$ and P is the insertion tableau of $\mathtt{Hecke}(w)$. Then the first row of P is given by $w_{r(w,1)}, w_{r(w,2)}, \ldots, w_{r(w,s)}$.*

Proof. By induction on n. The base case $n = 1$ is trivial.

Suppose that the claim holds for $w^\circ := w_1 w_2 \cdots w_{n-1}$. Thus if P° is the insertion tableau of $\mathtt{Hecke}(w^\circ)$ then by induction, the first row is given by

$$w^\circ_{r(w^\circ,1)}, w^\circ_{r(w^\circ,2)}, \ldots, w^\circ_{r(w^\circ,s^\circ)},$$

where $s^\circ = \mathtt{LIS}(w^\circ)$.

We consider the possibilities of what happens as we insert w_n into the first row of P°. We prove the desired conclusion holds for the case that $w^\circ_{r(w^\circ,t)} < w_n$ for some maximally chosen $t < s^\circ$; other cases are similar. So after inserting w_n, the first row of P is

$$w^\circ_{r(w^\circ,1)}, w^\circ_{r(w^\circ,2)}, \ldots, w^\circ_{r(w^\circ,t)}, w_n, w^\circ_{r(w^\circ,t+2)}, \ldots, w^\circ_{r(w^\circ,s^\circ)}.$$

The assumption shows that any longest increasing subsequence ω in w ending at w_n is of length at least $t + 1$ since we can adjoin w_n to the length t subsequence ending at $w^\circ_{r(w^\circ,t)}$. On the other hand, ω is of length at most $t+1$ since if it were of length $t+k+1$ for some $k>0$, there must be a length $t+k$ increasing subsequence in w° ending at w_r with $w_r < w_n$. Also $r \leqslant r(w^\circ, t+k)$ by the definition of $r(w^\circ, t+k)$. However, actually $r \neq r(w^\circ, t+k)$ since $w_{r(w^\circ,t+k)} \geqslant w_n$ by our choice of t. Hence $r < r(w^\circ, t+k)$. But this is a contradiction of the definition of $r(w^\circ, t+k)$ since we would then have a length $t+k+1$ increasing subsequence ending at $w^\circ_{r(w^\circ,t+k)} (\geqslant w_n)$.

Since we have just shown $w_n = w_{r(w,t+1)}$, it is now clear that $w_{r(w,h)} = w^\circ_{r(w^\circ,h)}$ for $h \neq t+1$. Thus, the first row of P satisfies the desired claim, and the induction follows. $\square$

In order to prove that the first column of λ computes $\texttt{LDS}(w)$, we need to draw a connection to [24], where we developed a K-theoretic *jeu de taquin* theory. Rather than repeat the setup in full here, for brevity, we refer the reader to that paper for the complete background on K-$\texttt{rectification}$ and K-$\texttt{infusion}$ used below.

Although what follows also constitutes a proof of our $\texttt{LIS}$ claims, we felt that including the direct proof via the stronger claim of Proposition 4.1 was worthwhile. However, a similarly direct proof of our $\texttt{LDS}$ claim seems harder.

Let

$$\gamma_n = (n, n-1, n-2, \ldots, 3, 2, 1)$$

be the staircase shape. Also, let $\lambda_{\mathrm{perm}(n)} = \gamma_n/\gamma_{n-1}$ be the **permutation shape** consisting of n single boxes arranged along an antidiagonal. Given $w \in W_{n,q}$, define $T_w \in \texttt{INC}(\lambda_{\mathrm{perm}(n)})$ to be the tableau where $w_1, w_2, \ldots, w_n$ is arranged from southwest to northeast. Also let $S \in \texttt{SYT}(\gamma_{n-1})$ be the **superstandard** Young tableau, i.e., the one whose first row is labeled $1, 2, \ldots, n-1$, the second row is labeled by $n, n+1, n+2, \ldots, 2n-3$, etc. This latter tableau determines a *particular* K-$\texttt{rectification}$ of T_w, which by definition is K-$\texttt{infusion}_1(S, T_w)$, see [24, Section 3]. (An important subtlety in K-theoretic jeu de taquin is that K-$\texttt{rectification}$ depends on the order in which it is performed, unlike the rectification of classical jeu de taquin. However, the order defined by S is particularly nice.)

The following result is an analogue of a classical result linking the Robinson–Schensted algorithm to the (ordinary) rectification of T_w:

Theorem 4.2. *Let $w \in W_{n,q}$. Then K-$\texttt{infusion}_1(S, T_w)$ is the insertion tableau of $\texttt{Hecke}(w)$.*

Proof. We induct on n. The base cases $n = 1, 2$ are easy. We may assume that the steps of the K-$\texttt{infusion}_1$ that are defined by the "inner" labels $\underline{n}, \underline{n+1}, \underline{n+2}, \ldots, \underline{\binom{n}{2}}$ results in a skew shape of the form $P^\circ \star \boxed{w_n}$, as depicted below:

$$\begin{array}{|c|c|c|c|c|}
\hline
\underline{1} & \underline{2} & \cdots & \underline{n-1} & w_n \\
\hline
P^\circ_{1,1} & P^\circ_{1,2} & \cdots & P^\circ_{1,n-1} \\
\cline{1-4}
P^\circ_{2,1} & \cdots & P^\circ_{2,n-2} \\
\cline{1-3}
\cdots & P^\circ_{n-2,2} \\
\cline{1-2}
P^\circ_{n-1,1} \\
\cline{1-1}
\end{array}. \qquad (12)$$

The induction hypothesis is that P° is the insertion tableau obtained by $\texttt{Hecke}$ inserting $w_1 w_2 \cdots w_{n-1}$. (In the depiction of P° from (12), some of the boxes with labels $P^\circ_{i,j}$ may be empty.) The non-underlined labels occupy the boxes of $P^\circ \star \boxed{w_n}$, whereas the underlined labels dictate the remaining steps to perform to complete the K-$\texttt{infusion}_1$ computation (these steps are recalled below).

Hence it remains to show that the tableau obtained by the $\texttt{Hecke}$-insertion $P^\circ \leftarrow w_n$ is the same as carrying out the K-$\texttt{infusion}_1$ indicated by (12), i.e., the operation

$$K\text{-}\texttt{infusion}_1\left(\begin{ytableau} \underline{1} & \underline{2} & \cdots & \underline{n-1} \end{ytableau},\ P^\circ \star \boxed{w_n}\right). \tag{13}$$

To do this, we first develop a technical fact. In [24, Section 1.1], we defined the procedure $\texttt{switch}$, which we restate now (in a more convenient form). Let $\texttt{Mixedtab}(\alpha, p, q)$ be the set of **mixed tableaux**, which, by definition, are tableaux of shape α, each of whose boxes is filled with an entry from one of two alphabets, $\{\underline{1}, \ldots, \underline{p}\}$ and $\{1, \ldots, q\}$, such that, within each row or column, the entries for each alphabet appear at most once. (No increasingness condition is demanded.) We also include the **null tableau** $\emptyset$, as a special element of $\texttt{Mixedtab}(\alpha, p, q)$.

Define an operator

$$\texttt{switch}(\underline{i}, j) : \texttt{Mixedtab}(\alpha, p, q) \to \texttt{Mixedtab}(\alpha, p, q)$$

as follows. Given $\emptyset \neq T \in \texttt{Mixedtab}(\alpha, p, q)$, consider the subshape S of T consisting of boxes whose entry is either $\underline{i}$ or j. For each non-singleton connected component of S, interchange the $\underline{i}$'s and the j's. If this results in a (non-null) mixed tableau, then the result is that tableau. Otherwise the result is $\emptyset$. By definition $\texttt{switch}(\underline{i}, j)(\emptyset) = \emptyset$.

Example 4.3. Let $\alpha = (4, 3, 1)$ and $p = q = 3$. Then $T \in \texttt{Mixedtab}(\alpha, p, q)$ is given below, together with two different $\texttt{switch}$ computations applied to it:

$$T \in \begin{ytableau} \underline{2} & \underline{1} & \underline{3} & 1 \\ \underline{1} & 3 & 1 \\ 2 \end{ytableau}, \quad \texttt{switch}(\underline{1}, 2)(T) = \begin{ytableau} \underline{2} & \underline{1} & \underline{3} & 1 \\ 2 & 3 & 1 \\ \underline{1} \end{ytableau}, \qquad \texttt{switch}(\underline{3}, 1)(T) = \begin{ytableau} \underline{2} & \underline{1} & 1 & \underline{3} \\ \underline{1} & 3 & \underline{3} \\ 2 \end{ytableau}.$$

On the other hand,

$$\texttt{switch}(\underline{1}, 2)\left(\begin{ytableau} 2 & \underline{1} \\ \underline{3} \\ \underline{1} \end{ytableau}\right) = \emptyset.$$

The following lemma is easy to verify from the definitions:

Lemma 4.4. *If $i \neq j$ and $r \neq s$ then the operators $\texttt{switch}(\underline{i}, r)$ and $\texttt{switch}(\underline{j}, s)$ commute, i.e.,*

$$\texttt{switch}(\underline{i}, r)\ \texttt{switch}(\underline{j}, s) \equiv \texttt{switch}(\underline{j}, s)\ \texttt{switch}(\underline{i}, r)$$

is a relation in the algebra generated by $\texttt{switch}$ operators on $\texttt{Mixedtab}(\alpha, p, q)$.

The procedure described in [24, Section 3] for computing K-$\texttt{infusion}_1(A, B)$ is to consider the entries of A as being underlined, where the maximum entry of A is $\underline{p}$, say, and the entries of B as not underlined, where the maximum entry of B is q. Now perform the $\texttt{switch}$ operations indexed by the following sequence of ordered pairs, read from left to right:

$$(\underline{p}, 1), (\underline{p}, 2), \ldots, (\underline{p}, q), (\underline{p-1}, 1), \ldots, (\underline{p-1}, q), \ldots, \ldots, (\underline{1}, 1), \ldots, (\underline{1}, q). \tag{14}$$

We refer to this sequence of pairs (interchangeably, the corresponding sequence of `switch` operators) as the **standard** `switch` **sequence**.

The technical fact we need is that K-`infusion` can in fact be computed differently: a `switch` sequence is called **viable** if it is a "shuffling" of (14), in the following sense:

- every $(\underline{i}, j)$ occurs exactly once, for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$;
- for any $1 \leqslant i \leqslant p$, the pairs $(\underline{i}, 1), \ldots, (\underline{i}, q)$ occur in that relative order; and
- for any $1 \leqslant j \leqslant q$ the pairs $(\underline{p}, j), \ldots, (\underline{1}, j)$ occur in that relative order.

This definition is explained by the proof of the following proposition:

Proposition 4.5. *Any viable* `switch` *sequence can be used to calculate* K-`infusion`.

Proof. It is straightforward to show that one can obtain any viable `switch` sequence from the standard `switch` sequence (14) by repeated applications of the commutation relation of Lemma 4.4. $\square$

Thus, in view of Proposition 4.5, to complete the induction it suffices to construct a viable `switch` sequence whose result is the same as $P^\circ \leftarrow w_n$. (A caution: in [24] it was shown that the standard `switch` sequence necessarily maintains increasingness along rows and columns of the members of each alphabet. We will not prove that a viable `switch` sequence also achieves this during the intermediate steps of a K-`infusion`. However, this does not play a logical role in how we apply Proposition 4.5 below.)

Let $y_1 := w_n$, and for $i > 1$, let y_i be the number which is inserted in row i according to `Hecke` insertion of w_n into P°. During a bumping step of `Hecke` insertion, let z_i be the smallest number already in row i that is greater than y_i.

We say that a mixed tableau, obtained after some number of `switch` operations applied to $P^\circ \star \boxed{w_n}$, is in **row** ***i*** **active form** if:

- all non-underlined symbols in rows $i+1$ and below have not moved from their initial positions,
- the i-th row is of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \underline{1} & \underline{2} & \cdots & \underline{k-1} & y_i & \underline{k} & \cdots & \underline{t} \\ \hline \end{array},$$

  where y_i has not yet moved from its initial position in column k,
- all non-underlined symbols in rows $i-1$ and above agree with the corresponding rows of `Hecke` insertion, with the possible exception that if y_i appears in row $i-1$ and column k of the `Hecke` insertion, then in row $i-1$ and column k of the mixed tableaux we have a $\underline{k-1}$.

Informally, the final condition in the definition of row i active form says that all rows above row i agree with the corresponding rows of `Hecke` insertion, except that there could be one "flaw" in row $i-1$ (where a $\underline{k-1}$ appears in the mixed tableau, while a y_i appears in the corresponding position in the `Hecke` insertion). In what follows, we show that this flaw, if it exists, will always be corrected and a y_i will indeed be placed there.

We say that a mixed tableau is in **row** ***i*** **terminated form** if

(1) all non-underlined symbols in rows $i+1$ and below have not moved from their initial positions,
(2) the i-th row is of the form

$$\begin{array}{|c|c|c|c|c|}\hline \underline{1} & \underline{2} & \cdots & \cdots & \underline{t} \\ \hline \end{array},$$

(3) the non-underlined symbols in rows $i-1$ and above agree with the corresponding rows of `Hecke` insertion.

We now show that, if `Hecke` insertion of w_n ends at the j-th row, then we can inductively construct a `switch` sequence which takes $P^\circ \star \boxed{w_n}$ from row 1 active form (its initial condition) through row i active form for $1 \leqslant i \leqslant j$, and then through row i terminated form for each $i > j$. More precisely, we will describe a sequence of `switches` which take a mixed tableau from row i active form to row $i+1$ active/terminated form. We then show how to put these sequences together to form a viable `switch sequence`. The result then follows.

Having explained our general strategy, what remains is some tedious but straightforward case analysis to describe the viable `switch` sequence we use. The base case is clear. Suppose we have arrived, after some sequence of applications of (A), (B), (A′), (B′) and (C′) below, at a mixed tableau in row i active form.

First suppose that y_i does not appear in row $i-1$ of the result of `Hecke` insertion. We therefore know (by the assumption that our mixed tableau is in row i active form) that its non-underlined entries in rows $i-1$ and higher, agree with the corresponding rows of the `Hecke` insertion. There are now two possibilities, (A) and (B), depending on what happens during the `Hecke` insertion of y_i into the i-th row.

(A) *y_i is inserted, bumping z_i*: Consider the `switch` sequence that moves y_i to the left along row i: specifically, using

$$\texttt{switch}(\underline{k-1}, y_i),\ \texttt{switch}(\underline{k-2}, y_i),\ \texttt{switch}(\underline{k-3}, y_i), \ldots, \tag{15}$$

until it is directly above the z_i in row $i+1$ of the mixed tableau, and then, starting from the right, swaps each box in row i having an underlined label with the one directly below, which has a non-underlined label (this can always be done since no label numerically equal to y_i appears among the latter boxes, by assumption). The result is therefore in row $i+1$ active form since z_i doesn't move in this process, it is the unique box with a non-underlined label in row $i+1$, and $y_{i+1} = z_i$, as demanded.

Note that the non-underlined labels of the i-th row of the mixed tableau we obtain after this process, agree with the i-th row of P°, with z_i replaced by y_i, as desired.

Example 4.6. If $i = 1$ and we started with

$$\begin{array}{|c|c|c|c|c|}\hline \underline{1} & \underline{2} & \underline{3} & \underline{4} & 3 \\ \hline 1 & 2 & 4 & 5 \\ \cline{1-4} 2 & 3 & 5 & 6 \\ \cline{1-4} \end{array}$$

so that $y_i = 3$ and $z_i = 4$, then we begin by moving y_i above $z_i = y_{i+1}$:

$$\begin{array}{|c|c|c|c|c|}\hline \underline{1} & \underline{2} & 3 & \underline{3} & \underline{4} \\ \hline 1 & 2 & 4 & 5 \\ \cline{1-4} 2 & 3 & 5 & 6 \\ \cline{1-4} \end{array}.$$

We conclude with `switch`$(\underline{3}, 5)$, `switch`$(\underline{2}, 2)$ and then finally `switch`$(\underline{1}, 1)$, resulting in

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & \underline{4} \\ \hline \underline{1} & \underline{2} & 4 & \underline{3} \\ \cline{1-4} 2 & 3 & 5 & 6 \\ \cline{1-4} \end{array},$$

which is in row $i+1(=2)$ active form. Moreover, the non-underlined labels in row $i = 1$ of this mixed tableau, namely $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline \end{array}$, agree with the first row of $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline \end{array} \leftarrow y_i$, as claimed.

(B) *y_i does not bump z_i when* `Hecke` *inserted into row i because y_i already appears in row i*: Proceed as in (A) by moving y_i to the left, until it is directly above z_i, i.e., apply (15). Now, "locally" the situation in the column containing z_i and the column to its left is

$$\cdots \begin{array}{|c|c|}\hline \underline{t} & y_i \\ \hline y_i & z_i \\ \hline \end{array} \cdots$$

for some t. To the right of the column containing z_i, we swap boxes with underlined and non-underlined labels, as in (A). Then we perform the transformation

$$\cdots \begin{array}{|c|c|}\hline \underline{t} & y_i \\ \hline y_i & z_i \\ \hline \end{array} \cdots \mapsto \cdots \begin{array}{|c|c|}\hline y_i & \underline{t} \\ \hline \underline{t} & z_i \\ \hline \end{array} \cdots. \tag{16}$$

After this transformation, we complete by swapping, right to left, the boxes to the left of the y_i, as in (A). The result is in the demanded row $i+1$ active form. (Note that there is an underlined entry in row i: this is because $y_{i+1} = z_i$ also appears as an entry in row i, at the position now occupied by $\underline{t}$ in the mixed tableau.)

Example 4.7. If $i = 1$ and we started with

$$\begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \underline{4} & 2 \\ 1 & 2 & 3 & 4 & \\ 4 & & & & \end{array}$$

then moving the "2" in row 1 to the left, as in (A), gives

$$\begin{array}{ccccc} \underline{1} & \underline{2} & 2 & \underline{3} & \underline{4} \\ 1 & 2 & 3 & 4 & \\ 4 & & & & \end{array}.$$

The remaining swaps give

$$\begin{array}{ccccc} \underline{1} & \underline{2} & 2 & \underline{3} & \underline{4} \\ 1 & 2 & 3 & 4 & \\ 4 & & & & \end{array} \mapsto \begin{array}{ccccc} \underline{1} & \underline{2} & 2 & 4 & \underline{4} \\ 1 & 2 & 3 & \underline{3} & \\ 4 & & & & \end{array} \mapsto \begin{array}{ccccc} \underline{1} & 2 & \underline{2} & 4 & \underline{4} \\ 1 & \underline{2} & 3 & \underline{3} & \\ 4 & & & & \end{array} \mapsto \begin{array}{ccccc} 1 & 2 & \underline{2} & 4 & \underline{4} \\ \underline{1} & \underline{2} & 3 & \underline{3} & \\ 4 & & & & \end{array}, \tag{17}$$

and the latter is in row $i + 1 = 2$ active form.

We now consider the case where y_i appears in the $(i-1)$-th row of the `Hecke` insertion. In this case, there are three possible ways for us to proceed, (A′), (B′), and (C′), depending on what happens during the `Hecke` insertion of y_i into row i. The first two possibilities, (A′) and (B′), parallel those already considered.

Note that in this case, the y_i in the $(i-1)$-th row of the `Hecke` insertion must be in column k, i.e., in the same column as the current position of y_i in row i of the mixed tableau. By the definition of row i active form, the mixed tableau has a $\underline{k-1}$ in row $i-1$ immediately above the y_i in row i.

(A′) *y_i is inserted, bumping z_i*: We apply the same strategy as in (A) above.

Note that in this case, the position at which y_i is inserted in row i by `Hecke`, must be strictly to the left of the position of y_i in row $i-1$ of the `Hecke` insertion. Thus, in applying our strategy from (A), we will begin by moving y_i to the left, and, at the first such `switch`, y_i will also move into the "flaw" in row $i-1$. After that point, row $i-1$ agrees with the result of `Hecke` insertion, and the rest of the analysis proceeds as in (A) above.

Example 4.8. The mixed tableau resulting from the sequence of `switches` (17) is in row 2 active form. The `Hecke` insertion of 3 into row 2 falls into case (A′). To obtain row 3 active form we have the further `switches`:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \underline{2} & 4 & \underline{4} \\ \hline \underline{1} & \underline{2} & 3 & \underline{3} \\ \cline{1-4} 4 \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & \underline{4} \\ \hline \underline{1} & 3 & \underline{2} & \underline{3} \\ \cline{1-4} 4 \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & \underline{4} \\ \hline 3 & \underline{1} & \underline{2} & \underline{3} \\ \cline{1-4} 4 \\ \cline{1-1}\end{array}$$

which puts us in row 3 active form. The "flaw" in row $2-1=1$ is fixed by the first `switch`.

(B′) *y_i does not bump z_i when* `Hecke` *inserted into row i because y_i appears in row i*: There are two possibilities. If in the mixed tableau the location of z_i (in row $i+1$) is strictly to the left of the position of the y_i (in row i), then we proceed as in (B); as in (A′) above, this also has the effect of correcting the "flaw" in row $i-1$. Otherwise, we are in the following situation:

$$\cdots\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \underline{k-1} \\ \hline \underline{k-1} & y_i \\ \hline y_i & z_i \\ \hline\end{array}\cdots.$$

We first `switch` the underlined entries to the right of y_i in row i (the middle row in the above depiction) with the non-underlined entries below them. We then perform the following `switch`:

$$\cdots\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \underline{k-1} \\ \hline \underline{k-1} & y_i \\ \hline y_i & z_i \\ \hline\end{array}\cdots \mapsto \cdots\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & y_i \\ \hline y_i & \underline{k-1} \\ \hline \underline{k-1} & z_i \\ \hline\end{array}\cdots.$$

This corrects the "flaw" in row $i-1$. We now finish as in (B).

(C′) *y_i does not replace z_i when Hecke inserted into row i because this would result in two vertically adjacent y_i entries, and* (B′) *does not hold*: Let b denote the entry to the left of z_i. Since z_i is the smallest entry in row i of P° which is greater than y_i, we know that $b \leqslant y_i$. The fact that (B′) does not hold implies that $b < y_i$.

Begin by `switch`ing all the boxes with underlined labels in row i to the right of y_i, with the non-underlined labels directly below them. The next `switches`, which involve the rows $i-1$ to $i+1$, in the column of y_i and the column to its immediate left, are given as follows

$$\cdots\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \underline{t} \\ \hline \underline{t} & y_i \\ \hline b & z_i \\ \hline\end{array}\cdots \mapsto \cdots\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \underline{t} \\ \hline b & y_i \\ \hline \underline{t} & z_i \\ \hline\end{array}\cdots \mapsto \cdots\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & y_i \\ \hline b & \underline{t} \\ \hline \underline{t} & z_i \\ \hline\end{array}\cdots. \tag{18}$$

As in (B), we finish by completing a sequence of `switches` involving the columns to the left of the b. The result of this process is a tableau in row $i+1$ active form.

As in the conclusion of (B), row i of the resulting row $i+1$ active form mixed tableau, contains an underlined label, since $z_i(=y_{i+1})$ appears in row i of the `Hecke` insertion tableau, in the position corresponding to the current position of $\underline{t}$ in row i of the mixed tableau.

Example 4.9. To give an example of (C′), the previous insertion must have been of type (B), (B′), or (C′), so consider the following example:

$$\begin{array}{|c|c|c|c|}\hline \underline{1} & \underline{2} & \underline{3} & 3 \\ \hline 1 & 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 4 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}.$$

After the first step, an insertion of type (B), we reach row 2 active form:

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & \underline{2} & \underline{3} \\ \hline \underline{1} & \underline{2} & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 4 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}.$$

As compelled by the conditions of a viable `switch` sequence, we `switch` $\underline{2}$ with 4 before we switch $\underline{2}$ with 5, and as a result, as described above, we get to row 3 active form:

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & \underline{3} \\ \hline 2 & 4 & \underline{2} & \multicolumn{1}{c}{} \\ \cline{1-3} \underline{1} & \underline{2} & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}.$$

The final result is

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & \underline{3} \\ \hline 2 & 4 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} 6 & \underline{1} & \underline{2} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}.$$

Again, this tableau agrees with $P^\circ \leftarrow 3$.

Using similar analysis one can give `switch` sequences for the terminating steps of `Hecke` insertion, such that one obtains row i terminating form for all $j < i < \ell + 1$ where ℓ is the number of rows of P°. We leave the straightforward details to the reader.

These constructions then show, by induction on the number of rows ℓ, that we have a sequence of `switch` operations transforming $P^\circ \star \boxed{w_n}$ into $P^\circ \leftarrow w_n$.

Conclusion of the proof of Theorem 4.2: by the fact that P° is an increasing tableau, and by the definition of row i active/terminating forms, it is easy to see that the sequence of `switch` operations used forms a viable sequence, after suitable insertions of any trivial `switch`$(\underline{i}, r)$ operators (that is to say, switch operations which do not have any effect on the tableau). We can therefore apply Proposition 4.5 as we claimed earlier. □

Example 4.10. Continuing Example 4.9, the `switch` sequence we obtain, by following the descriptions of the cases (B) and (C) that are needed is:

$$(\underline{3}, 2), (\underline{2}, 3), (\underline{2}, 4), (\underline{2}, 5), (\underline{2}, 6), (\underline{1}, 6).$$

This is not quite a viable sequence: although our constructions guarantee that it satisfies the second and third conditions to be a viable sequence, it fails the first, since, e.g., $(\underline{3}, 1)$ doesn't appear in the sequence, since this `switch` is never needed. However, clearly we can simply insert this trivial `switch`, along with the others that are missing, giving the viable sequence:

$$(\underline{3},1),(\underline{\mathbf{3}},\mathbf{2}),(\underline{3},3),(\underline{3},4),(\underline{3},5),(\underline{3},6),(\underline{2},1),(\underline{2},2),(\underline{\mathbf{2}},\mathbf{3}),(\underline{\mathbf{2}},\mathbf{4}),(\underline{\mathbf{2}},\mathbf{5}),(\underline{\mathbf{2}},\mathbf{6}),$$
$$(\underline{1},1),(\underline{1},2),(\underline{1},3),(\underline{1},4),(\underline{1},5),(\underline{\mathbf{1}},\mathbf{6}).$$

The action of this viable sequence on the original mixed tableau is therefore the same as the original switch sequence, which we highlight in boldface. This viable sequence also happens to be the standard switch sequence, although it needn't be in general. Hence

$$K\text{-}\mathtt{infusion}_1\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,,\ \begin{array}{cccc}\cline{4-4} &&&\multicolumn{1}{|c|}{3}\\\hline \multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{5}&\\\cline{1-3} \multicolumn{1}{|c|}{2}&\multicolumn{1}{c|}{4}&\multicolumn{1}{c|}{6}&\\\cline{1-3}\end{array}\right)=\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}\leftarrow 3,$$

in agreement with Theorem 4.2.

In [24, Theorem 6.1] we showed that the first row of $K\text{-}\mathtt{infusion}_1(R,T_w)$ has length $\mathtt{LIS}(w)$, for any increasing tableau R of shape γ_{n-1}. So by Theorem 4.2, the first row of $\mathtt{Heckeshape}(w)=K\text{-}\mathtt{infusion}_1(S,T_w)$ has length $\mathtt{LIS}(w)$. By symmetry, [24, Theorem 6.1] also implies that the first column of $K\text{-}\mathtt{infusion}_1(R,T_w)$ has length $\mathtt{LDS}(w)$. Hence the LDS claim follows. This completes the proof of Theorem 1.3.

Given $w=w_1w_2\cdots w_n$, define $\mathtt{rev}(w)=w_nw_{n-1}\cdots w_1$. The following is symmetry statement is immediate from Theorem 1.9, since $\mathtt{LIS}(w)=\mathtt{LDS}(\mathtt{rev}(w))$:

Corollary 4.11. *Let* $\lambda=\mathtt{Heckeshape}(w)$ *and* $\mu=\mathtt{Heckeshape}(\mathtt{rev}(w))$. *Then* $\lambda_1=\mu_1'$ *and* $\mu_1=\lambda_1'$ *where* λ' *and* μ' *are the conjugate shapes of* λ *and* μ *respectively.*

A warning is needed: unlike Robinson–Schensted correspondence setting, with Hecke, one *cannot* conclude that the insertion tableaux associated to $w=w_1w_2\cdots w_n$ and $\mathtt{rev}(w)=w_nw_{n-1}\cdots w_1$ differ only by a reflection across the main diagonal. A counterexample is $w=1\ 3\ 4\ 2\ 2$. (In [20], the symmetry property of the Robinson–Schensted correspondence, was applied to prove the LDS claim in the classical version of Theorem 1.3.)

Problem 4.12. Give an explicit description of $\mathtt{Hecke}(\mathtt{rev}(w))$ in terms of $\mathtt{Hecke}(w)$.

Finally, Greene [12] has given an explanation of the other rows of the shape λ associated to a permutation $w\in S_n$ under the Robinson–Schensted correspondence: $\lambda_1+\cdots+\lambda_i$ equals the maximal size of a union of i disjoint increasing subsequences of w.

However, we could not find any extension of Greene's theorem in the Hecke context. The naive attempts do not work: Since $|\lambda|\leqslant n$, the simplest case to analyze is when $|\lambda|=n$. The example $w=2\ 1\ 2\ 3\ 2$ corresponds to $\lambda=(3,2)$; this shows that it is not valid to merely replace "increasing" by "strictly increasing" in Greene's theorem, since that would predict $\lambda=(3,1,1)$.

5. Probabilistic combinatorics and proofs of Theorems 1.8 and 1.9

Proof of Theorem 1.8. Let $w_0=\left(\begin{smallmatrix}1&2&3&\ldots&q-1&q&q+1\\ q+1&q&q-1&\ldots&3&2&1\end{smallmatrix}\right)$ be the word in S_{q+1} of maximal Coxeter length. Hence $\ell(w_0)=\binom{q+1}{2}$. This is the unique permutation in S_{q+1} with this length.

We need the following lemma, which characterizes when $\mathtt{Heckeshape}(w)$ is maximized.

Lemma 5.1. *For* $w\in W_{n,q}$, $\mathtt{Heckeshape}(w)=(q,q-1,\ldots,3,2,1)$ *if and only if* $W(w)=w_0$.

Proof. First suppose $W(w)=w_0$. Under $\mathtt{Hecke}(w)$, the insertion tableau P satisfies $W(\mathtt{word}(P))=w_0$. Hence the shape of P has at least $\binom{q+1}{2}$ boxes and so by Proposition 1.2 it must be $(q,q-1,\ldots,3,2,1)$.

Conversely, if $\texttt{Heckeshape}(w) = (q, q-1, \ldots, 3, 2, 1)$, then note that there is a unique increasing filling P of that shape (using $1, 2, \ldots, q$ in the first row, $2, 3, \ldots, q$ in the second row, etc.). Then it is well known that $W(\texttt{word}(P)) = w_0$. □

In view of Lemma 5.1, the theorem will follow if we can show that

$$W(w) = w_0 \quad \text{almost surely, as } n \to \infty. \tag{19}$$

(We conjecture this to be true whenever $0 \leqslant \alpha < \alpha_{\text{critical}} = \frac{1}{2}$. This would imply Conjecture 1.6 for this entire range.)

Set $w(k) := w_1 \cdots w_k$. Then $\ell(W(w(k)w_{k+1})) = \ell(W(w(k))) + 1$ if the simple reflection $W(w_{k+1})$ (say equal to $s_t = (t\ t+1)$) is an **ascent** of $W(w(k))$ and $\ell(W(w(k)w_{k+1})) = \ell(W(w(k)))$ otherwise. (An **ascent** occurs at a position t for a permutation π if $\pi(t) < \pi(t+1)$.)

Provided that $\pi \neq w_0$, π has at least one ascent. Thus when $W(w(k)) \neq w_0$, the probability that w_{k+1}'s introduction increases the Coxeter length is at least $\frac{1}{q}$.

Let

$$E_k = \text{the event that } \ell\bigl(W\bigl(w(k)\bigr)\bigr) < \tbinom{q+1}{2}.$$

Related to this, let $Y_i \in \{0, 1\}$ be Bernoulli distributed with parameter $\frac{1}{q}$ (i.e, $Y_i = 1$ with probability $\frac{1}{q}$ and $Y_i = 0$ with probability $1 - \frac{1}{q}$). Set

$$Z_k := Y_1 + \cdots + Y_k.$$

Clearly,

$$Prob(E_k) \leqslant Prob\left(Z_k < \binom{q+1}{2}\right). \tag{20}$$

We now show that when

$$k = O\left(q^{3+\epsilon}\right), \quad \text{for } 0 < \epsilon,$$

the right-hand side of the inequality (20) goes to zero as $q \to \infty$.

This is a simple application of (a special case of) Bennet's large deviation inequality, see, e.g., [8, Corollary 2.4.7]: suppose X_i are independent, mean zero random variables with $|X_i| \leqslant 1$. Set $S_k = \sum_{i=1}^{k} X_i$. Then for $y \geqslant 0$ we have

$$Prob\bigl(k^{-\frac{1}{2}} S_k \geqslant y\bigr) \leqslant e^{-\frac{y^2}{2}}. \tag{21}$$

To apply this to our setting, let $X_i = -Y_i + \frac{1}{q}$. Hence $S_k = -Z_k + \frac{k}{q}$. Then with $a = o(q^2)$

$$\begin{aligned} Prob(Z_k < a) &= Prob\left(-S_k < a - \frac{k}{q}\right) = Prob\left(S_k \geqslant \frac{k}{q} - a\right) \\ &\leqslant Prob\left(S_k > \frac{k}{2q}\right) \quad \bigl(\text{for } q \text{ large, since } a = o\bigl(q^2\bigr)\bigr) \\ &= Prob\left(k^{-\frac{1}{2}} S_k \geqslant \frac{\sqrt{k}}{2q}\right) \leqslant e^{-\frac{k}{8q^2}} \to 0, \quad \text{as } q \to \infty. \end{aligned}$$

The result then follows. □

In the above argument, we interpreted $w \in W_{n,q}$ as a random walk in S_{q+1} that begins at the identity and works its way up in the weak Bruhat order to w_0. At each step the probability of going up is *at least* $\frac{1}{q}$ (as we have used), but is larger in general. However, since this probability varies, even for permutations with the same Coxeter length, a more refined analysis is needed to push the argument we have used further, up towards $\alpha_{\text{critical}} = \frac{1}{2}$.

Proof of Theorem 1.9. For $0 \leqslant \alpha < \alpha_{\text{critical}} = \frac{1}{2}$ we will apply an argument similar to that for Theorem 1.8.

Given $u \in W_{n,q}$ let

$$m(u) = \max_{t \geqslant 1} 1, 2, \ldots, t \quad \text{is a subsequence of } u.$$

Let $w(k) = w_1 \cdots w_k$ and set

$$E_k = \text{the event that } m\big(w(k)\big) < q.$$

Provided E_k occurs, then

$$m\big(w(k)w_{k+1}\big) = m\big(w(k)\big) + 1$$

with probability $\frac{1}{q}$, and is equal to $m(w(k))$ otherwise.

Let $\{Y_i\}$ and $Z_k = Y_1 + \cdots + Y_k$ be discrete random variables, where Y_i is Bernoulli distributed with parameter $\frac{1}{q}$. Now,

$$Prob(E_k) = Prob(Z_k < q).$$

Thus it will be enough to show that when $k = O(q^{2+\epsilon})$ for $\epsilon > 0$ then

$$Prob\big(Z_k < O\big(q^{1+\epsilon}\big)\big) \to 0 \quad \text{as } q \to \infty.$$

This is another application of the large deviation inequality (21).

For $\alpha_{\text{critical}} = \frac{1}{2} < \alpha \leqslant 1$ we use a proof provided for us by O. Zeitouni: $E(\texttt{LwIS})$, the expected length of the longest weakly increasing subsequence of $w \in W_{n,q}$ (with $\alpha > \alpha_{\text{critical}} = \frac{1}{2}$) is known to satisfy

$$E(\texttt{LwIS}) \approx 2\sqrt{n};$$

see [15, Theorem 1.7]. The argument shows that the difference between the LIS and LwIS of w is typically small.

Let $\texttt{LwIS}_{a,b}$ be the random variable for the value of $\texttt{LwIS}(w)$ of a random uniform word $w \in W_{\lfloor a \rfloor, \lfloor b \rfloor}$ where $\lfloor a \rfloor$ is the integer part of a, etc. Similarly define $\texttt{LIS}_{a,b}$ where LIS replaces LwIS.

Fix $\epsilon > 0$ (unrelated to the ϵ used in the first half of the proof above) and let $L_0 = L_0(\epsilon)$ be large enough such that

$$\inf_{L > L_0} \underline{\lim}_{n \to \infty} Prob\big(\texttt{LwIS}_{L^2(1-\epsilon), \frac{qL}{\sqrt{n}}} > 2(1-4\epsilon)L\big) > 1 - \epsilon. \tag{22}$$

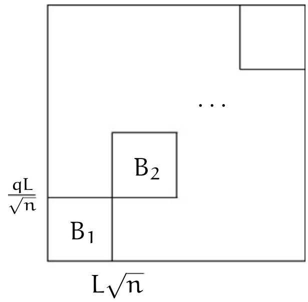

Fig. 4. $\alpha > \alpha_{\text{critical}} = \frac{1}{2}$ case of proof of Theorem 1.9.

We need a "graphical" representation of a word in $W_{n,q}$: consider a $q \times n$ rectangle subdivided into unit squares. In each of the n columns, one places a single "dot" in one of the q rows. The set of such configurations is in obvious bijection with words in $W_{n,q}$.

Given L, draw $\sqrt{n}/L$ smaller rectangles of dimension $\frac{qL}{\sqrt{n}} \times L\sqrt{n}$ along an antidiagonal inside the $q \times n$ rectangle, as depicted in Fig. 4 below.

Label the i-th southwest most box B_i. Let N_i be the random variable giving the number of dots in B_i. Notice that the N_i's are independent. Also, the dots inside B_i define a word, and we can speak of $\mathtt{LIS}(B_i)$ and $\mathtt{LwIS}(B_i)$, the length of the longest strictly (respectively, weakly) increasing subsequence of that word.

Say that B_i is **good** if the following conditions simultaneously hold:

(a) $N_i \geqslant L^2(1-\epsilon)$;
(b) $\mathtt{LwIS}(B_i) \geqslant 2(1-4\epsilon)L$; and
(c) no two dots in B_i have the same height (hence $\mathtt{LwIS}(B_i) = \mathtt{LIS}(B_i)$).

Now, we have

$$E(N_i) = L\sqrt{n} \times \left(\frac{qL}{\sqrt{n}} \times \frac{1}{q}\right) = L^2$$

and we claim for an L_1 sufficiently large, for $L \geqslant L_1$, and for all n large, we have

$$Prob\big(N_i \leqslant L^2(1-\epsilon)\big) \leqslant \epsilon.$$

The proof is a standard argument: let Y_k be the indicator random variable which evaluates to 1 if column k has a dot that lies in the box B_i that occupies that column (hence with probability $L/\sqrt{n}$), and evaluates to 0 otherwise. Hence

$$\begin{aligned}
Prob\big(N_i \leqslant L^2(1-\epsilon)\big) &= Prob\left(\sum_{k=1}^{L\sqrt{n}} Y_k \leqslant L^2(1-\epsilon)\right) \\
&= Prob\left(\sum_{k=1}^{L\sqrt{n}} \big(Y_k - E(Y_k)\big) < L^2(1-\epsilon) - L^2\right) \\
&= Prob\left(\sum_{k=1}^{L\sqrt{n}} \big(Y_k - E(Y_k)\big) < -L^2\epsilon\right) \\
&\leqslant \frac{L\sqrt{n} \cdot (L/\sqrt{n})}{L^4\epsilon^2} = \frac{1}{L^2\epsilon^2},
\end{aligned}$$

where the previous line is an application of Chebyshev's inequality. Now take L sufficiently large (bigger than some L_1) so $\frac{1}{L^2\epsilon^2} \leqslant \epsilon$.

Assuming $L \geqslant L_0$, if the event (a) occurs, then with probability at least $1-\epsilon$, when n is large, (b) holds, because of the definition of L_0 using (22).

The probability of the event (c) not occurring is bounded above (using a union bound) by

$$\begin{aligned}\#\text{rows} \times \mathit{Prob}(\text{two dots share a given height}) &\leqslant \frac{qL}{\sqrt{n}} \times (L\sqrt{n})^2 \times (1/q)^2 \\ &= L^3\frac{\sqrt{n}}{q} \to 0,\end{aligned}$$

because $n = o(q^2)$.

So,

$$\mathit{Prob}(B_i \text{ is good}) \geqslant 1 - 3\epsilon$$

for $L > \max(L_0, L_1)$ and n large.

Another standard argument with Chebyshev's inequality shows with high probability, say at least $1-\epsilon$, and for n large, the number of good boxes is

$$\frac{\sqrt{n}}{L}(1-\epsilon)(1-3\epsilon) \geqslant \frac{\sqrt{n}}{L}(1-4\epsilon).$$

Hence, with that probability, for $w \in W_{n,q}$

$$\mathtt{LIS}_{n,q} \geqslant \frac{\sqrt{n}}{L}(1-4\epsilon) \times 2(1-4\epsilon)L \geqslant 2\sqrt{n}(1-8\epsilon).$$

Since

$$2\sqrt{n}(1-8\epsilon) \leqslant E(\mathtt{LIS}_{n,q}) \leqslant E(\mathtt{LwIS}_{n,q}) \approx 2\sqrt{n}$$

the $\alpha > \alpha_{\text{critical}} = \frac{1}{2}$ case follows by taking $\epsilon \to 0$, completing the proof of the theorem. □

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Appendix A. (By A. Yong and O. Zeitouni)

The goal of this appendix is to present a proof of the following result:

Theorem A.1. *Let $q = k\sqrt{n} + o(\sqrt{n})$. Then*

$$E(\texttt{LIS}) \approx \beta(k)2\sqrt{n}$$

where

$$\beta(k) = \begin{cases} \frac{k}{2} & \text{if } 0 < k \leqslant 1, \\ \frac{2-k^{-1}}{2} & \text{if } k > 1. \end{cases}$$

However, in order to prove this statement, we need to work with another variant of Plancherel measure, utilized, e.g., by [3] and alluded to in Section 1.5 of the main text. Our approach parallels the one developed in [18,26] to prove $E(\texttt{LIS}) = 2\sqrt{n}$ in the permutation case, by utilizing work of [3].

A.1. Preliminaries

A **semistandard Young tableau** of shape $\lambda \in \mathbb{Y}$ with labels from $\{1, 2, \ldots, q\}$ is a filling of the Young shape λ with these labels so that the entries weakly increase along rows, and strictly increase along columns. For example, if $\lambda = (2, 1)$ and $q = 2$ there are two such tableaux: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}$. Let $g^{\lambda}(q)$ denote the number of such tableaux.

Define the **Plancherel-**RSK **measure** $\nu_{n,q}$ on the set $\mathbb{Y}_n$ of Young diagrams λ with n boxes, by declaring that a random Young shape $\lambda_{n,q}$ occurs with probability

$$Prob(\lambda_{n,q} = \lambda) = \frac{1}{q^n} f^{\lambda} g^{\lambda}(q).$$

We make no claims of originality in this definition. Indeed, this is the same measure studied in, e.g., [3]; although there the measure is defined in terms of dimensions of irreducible S_n and $GL_n(\mathbb{C})$ modules associated to λ; the equivalence is well known. The fact that $\nu_{n,q}$ is in fact a probability distribution follows from either Schur–Weyl duality, as in Section 1.5, or by the RSK algorithm, see, e.g., [22, Section 7.11].

A crucial advantage of $\nu_{n,q}$ for the purposes of understanding $E(\texttt{LIS})$, in comparison to Plancherel–Hecke measure, is that both f^{λ} and $g^{\lambda}(q)$ have simple multiplicative formulas. This makes it more readily analyzed using ideas of [18,26], which we modify to the present setting.

Given a box $u \in \lambda$, define the **hook-length** associated to u to be $H(u) := A(u) + L(u) + 1$ where $A(u)$ is the number of boxes strictly to the right of u, and in the same row, and $L(u)$ is the number of boxes strictly below u and in the same column. Then we have [22, Chapter 7], the **hook-length formula** and **hook-content formulas**, respectively:

$$f^{\lambda} = \frac{n!}{\prod_{u \in \lambda} H(u)} \quad \text{and} \quad g^{\lambda}(q) = \prod_{u \in \lambda} \frac{q + C(u)}{H(u)},$$

where in the second formula $C(u)$ is the **content** of u, the column index of u minus the row index of u. So for example, if $\lambda = (4, 3, 2)$, the contents are given by

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & \\ \cline{1-3} -2 & -1 & & \\ \cline{1-2}\end{array}.$$

A.2. Plancherel-RSK as a Markov measure

Young's lattice is the poset structure on $\mathbb{Y}$ where $\lambda \leqslant \mu$ if the shape of λ is contained in the shape of μ. We write $\lambda \to \mu$ to denote a covering relation in this poset, i.e., where μ is obtained from λ by adding a single box at a corner.

Define a Markov process on $\mathbb{Y}$ with the transition probabilities

$$Prob(\lambda \to \mu) = \frac{g_\mu(q)}{q g_\lambda(q)}.$$

We need the following lemma, that in particular shows that Plancherel-RSK measure is a Markov measure with the above transition probabilities.

Lemma A.2.

(I) $\sum_{\mu:\lambda\to\mu} Prob(\lambda \to \mu) = 1$.
(II) $\sum_{\lambda:\lambda\to\mu} Prob(\lambda \to \mu)\nu_{n,q}(\lambda) = \nu_{n+1,q}(\mu)$.

Proof. The claim (I) is equivalent to

$$\sum_{\mu:\lambda\to\mu} g^{\mu}(q) = q g^{\lambda}(q).$$

This follows from the following *Pieri rule* for Schur polynomials

$$\sum_{\mu:\lambda\to\mu} s_\mu(x_1,\dots,x_q) = s_{(1)}(x_1,\dots,x_q)\cdot s_\lambda(x_1,\dots,x_q).$$

See [22, Theorem 7.15.7]. Here

$$s_\lambda(x_1,\dots,x_q) = \sum_T \mathbf{x}^T$$

is the **Schur polynomial**, where the sum is over all semistandard Young tableaux of shape λ with entries from $\{1,2,\dots,q\}$, $\mathbf{x}^T = x_1^{i_1}x_2^{i_2}\cdots x_q^{i_q}$, and i_j is the number j's used in T. In particular, (I) is immediate from $g^\lambda(q) = s_\lambda(1,1,\dots,1)$.

For (II), the claim is

$$\sum_{\lambda:\lambda\to\mu}\left(\frac{g^\mu}{q g^\lambda}\right)\left(\frac{f^\lambda g^\lambda}{q^n}\right) = \frac{f^\mu g^\mu}{q^{n+1}},$$

that is, $\sum_{\lambda:\lambda\to\mu} f^\lambda = f^\mu$, which is well known (and straightforward from the definitions). $\square$

A.3. Conclusion of proof of Theorem A.1

Work of Biane [3, Theorem 3] describes the typical shape under Plancherel-RSK after the rescaling $\hat{f}(x) := \frac{1}{2\sqrt{n}} f(2\sqrt{n}x)$. Biane's theorem implies that

$$E(\mathtt{LIS}) \geqslant \beta(k) \tag{A.1}$$

but not

$$E(\mathtt{LIS}) \leqslant \beta(k). \tag{A.2}$$

Briefly, we explicate how his work applies to our situation (the reader is directed to the original source for details): Biane works with the coordinate axes rotated 45-degrees counterclockwise, as in, e.g., [26,27]. His aforementioned theorem states that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of (rescaled and rotated) Young diagrams, with f_n chosen according to the distribution $\nu_{n,q}$, then for any $\epsilon > 0$ we have

$$\lim_{n\to\infty} Prob\Big(\sup_{u\in\mathbb{R}}\big|f_n(u) - P_{\frac{1}{k}}(u)\big| > \epsilon\Big) \to 0. \tag{A.3}$$

Also, $P_{\frac{1}{k}}$ is Biane's limit shape, which has the property that it meets the line $y = -x$ at a distance $\beta(k)$ from the origin. In other words, the "first column" C of $P_{\frac{1}{k}}$ satisfies

$$C = \beta(k). \tag{A.4}$$

For each n, let C_n be the length of the first column of f_n. From (A.3) it follows that for any $\epsilon > 0$,

$$\lim_{n\to\infty} Prob(C_n < C - \epsilon) \to 0. \tag{A.5}$$

Moreover, since it is known that for any $w \in W_{n,q}$ the first column of the Young diagram associated to $\mathtt{RSK}(w)$ equals $\mathtt{LDS}(w)$, (A.1) follows immediately from (A.4) and (A.5) combined.

Note that the above argument does not also prove (A.2) since (A.3) does not rule out the possibility that $\{f_n\}_{n=1}^{\infty}$ consists of Young diagrams with "tails" along the $y = -x$ axis that both "lengthen" and "thin out" as $n \to \infty$. Therefore, it remains to verify (A.2).

To do this, we modify an argument found in [27], which establishes the analogous assertion in the permutation case: consider the set $\mathbb{Y}^{\infty}$ of all sequences of Young diagrams

$$\boldsymbol{\lambda} = \big(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \ldots, \lambda^{(i)}, \ldots\big)$$

where $\lambda^{(i)} \to \lambda^{(i+1)}$ for $i \geqslant 1$.

For a Young diagram λ, let $\lambda^{\downarrow}$ denote the diagram obtained by adding a single box to λ, in the first column. For each integer $i \geqslant 1$, define the indicator function $\psi_i : \mathbb{Y}^{\infty} \to \{0,1\}$ by setting $\psi_i(\boldsymbol{\lambda}) = 1$ if $\lambda^{(i)} = (\lambda^{(i-1)})^{\downarrow}$, and setting $\psi_i(\boldsymbol{\lambda}) = 0$ otherwise.

Studying the expectation of ψ_i with respect to measure on $\mathbb{Y}^{\infty}$ induced from the aforementioned Markov process, we have

$$\begin{aligned}
E(\psi_i)^2 &= \bigg(\sum_{\lambda} \nu_{i-1,q}(\lambda) \cdot Prob\big(\lambda \to \lambda^{\downarrow}\big)\bigg)^2 \\
&\leqslant \sum_{\lambda} \nu_{i-1,q}(\lambda) \cdot Prob\big(\lambda \to \lambda^{\downarrow}\big)^2 \quad \text{(Cauchy–Schwarz inequality)} \\
&= \sum_{\lambda} \nu_{i-1,q}(\lambda) \frac{f^{\lambda^{\downarrow}} g^{\lambda^{\downarrow}}(q)}{f^{\lambda} g^{\lambda}(q)} \cdot \frac{1}{q} \cdot \frac{1}{q} \cdot \frac{g^{\lambda^{\downarrow}}(q)/g^{\lambda}(q)}{f^{\lambda^{\downarrow}}/f^{\lambda}} \\
&= \frac{1}{q} \sum_{\lambda} \nu_{i,q}\big(\lambda^{\downarrow}\big) \frac{g^{\lambda^{\downarrow}}(q)/f^{\lambda^{\downarrow}}}{g^{\lambda}(q)/f^{\lambda}},
\end{aligned}$$

where we have just used

$$\nu_{i,q}\big(\lambda^{\downarrow}\big) = \nu_{i-1,q}(\lambda)\frac{f^{\lambda^{\downarrow}} g^{\lambda^{\downarrow}}(q)}{f^{\lambda} g^{\lambda}(q)} \cdot \frac{1}{q}.$$

Let

$$L(\lambda) := g^{\lambda}(q)/f^{\lambda}.$$

Note that by the hook-content formula we have

$$L\big(\lambda^{\downarrow}\big)/L(\lambda) = \frac{\prod_{u\in\lambda^{\downarrow}} \frac{q+c(u)}{i!}}{\prod_{u\in\lambda} \frac{q+c(u)}{(i-1)!}} = \frac{q-\lambda_1'}{i}$$

where λ_1' is the length of the first column of λ.

Summarizing, we have

$$E(\psi_i)^2 \leqslant \frac{1}{q}\sum_{\lambda} \nu_{i,q}\big(\lambda^{\downarrow}\big)\frac{q-\lambda_1'}{i} = \frac{1}{qi}(q-\gamma_i), \tag{A.6}$$

where γ_i denotes the expectation of $(\lambda_1^{(i)})'$, i.e., the expected length of the first column of a random shape with i boxes, drawn under the Plancherel-RSK measure.

Notice also that since ψ_i is an indicator random variable, we have

$$E\big(\psi_i^2\big) = E(\psi_i) = \gamma_i - \gamma_{i-1}. \tag{A.7}$$

Therefore, combining (A.6) and (A.7) we obtain, by the Cauchy–Schwarz inequality, the following difference inequality:

$$\gamma_i - \gamma_{i-1} \leqslant \sqrt{\frac{1}{qi}}\sqrt{q-\gamma_i}. \tag{A.8}$$

We claim that $\gamma_i \leqslant \beta(i)2\sqrt{n}$.

To prove this, note the following facts about γ_i:

(a) $\gamma_{i+1} \geqslant \gamma_i$; and
(b) $\gamma_i \leqslant q$.

Now define a linear interpolation: for $t \in [i/q, (i+1)/q]$, set

$$\beta_t = \frac{\gamma_i}{q} + q\left(t - \frac{i}{q}\right)\left(\frac{\gamma_{i+1}}{q} - \frac{\gamma_i}{q}\right). \tag{A.9}$$

Note that for such t, $\sqrt{1-\gamma_i/q} \leqslant \sqrt{1-\beta_t}$. Therefore, combining Eqs. (A.8) and (A.9) we obtain

$$\frac{d}{dt}\beta_t = \gamma_{i+1} - \gamma_i \leqslant \frac{1}{\sqrt{qt}}\sqrt{1-\beta_t}, \qquad \beta_{1/q} = 1/q. \tag{A.10}$$

Since $\gamma_i, \gamma_{i+1} \leqslant q$ we have $\beta_t \leqslant 1$, hence the above differential inequality is equivalent to

$$-\frac{d}{dt}\sqrt{1-\beta_t} \leqslant 1/(2\sqrt{qt}).$$

Hence it follows that

$$\sqrt{1-\beta_t} \geqslant \sqrt{1-1/q} - \sqrt{t/q} + 1/q \geqslant \sqrt{1-1/q} - \sqrt{t/q}.$$

That is,

$$\beta_t \leqslant 2\sqrt{t/q} - (t-1)/q.$$

Now we care about $t = n/q = \sqrt{n}/k$. We always have $\beta_{n/q} \leqslant 1$ (trivially), but for $k > 1$ we have the better inequality $\beta_{n/q} \leqslant 2/k - 1/k^2 + 1/q$. Therefore,

$$\gamma_n = q\beta_{n/q} \leqslant q,$$

for $k \leqslant 1$ and

$$\gamma_n = q\beta_{n/q} \leqslant (2 - 1/k)\sqrt{n} + 1,$$

for $k > 1$. The result then follows.

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